Complete Quadrilaterals: Exploring the Elegance of Geometry

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Abstract

This article explores the complex geometric structure of complete quadrilaterals, examining them through the lens of Menelaus theorem, harmonic points, Miquel points, Newton line, and Gauss-Bodenmiller theorem. The exposition is supplemented with an illustrative example.

Keywords: Complete quadrilateral, Menelaus Theorem, harmonic points, Miquel point, Newton line, Gauss-Bodenmiller Theorem

1 Introduction

When four non-parallel lines lie within the same plane, with no three lines intersecting at a single point, they form what is known as a complete quadrilateral (as depicted in Figure 1 below). This configuration comprises six vertices labeled as A, B, C, D, E, and F, along with three diagonals, BD, AE, and CF [1].

This expository essay will explore intriguing geometric structures of complete quadrilaterals through a number of notions and theorems.

2 Menelaus Theorem

In the vicinity of the ancient Aegean Sea, the Greeks dwelled. They developed a brilliant civilization and the following theorem was one of their results. It is attributed to Menelaus of Alexandria (c. 70 – 140 CE), a mathematician and astronomer [3].

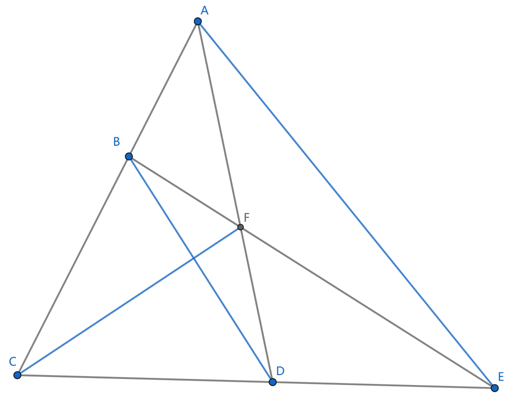

Figure 1: Complete quadrilateral

Theorem 2.1. Refer to Figure 1. Consider the triangle $\triangle ACD$ and the transversal line BFE. The following identity holds:

$$\frac{|AB|}{|BC|} \times \frac{|CE|}{|ED|} \times \frac{|DF|}{|FA|} = 1$$

where $|\cdot|$ denotes the length of the segment.

Proof: Note that AB and BC are the bases of triangles $\triangle ABF$ and $\triangle CBF$, respectively, and these triangles have a common height. Likewise, DF and FA are the bases of $\triangle DBF$ and $\triangle ABF$ that share a height. Thus, the ratio of the segment lengths can be replaced by the areas of the respective triangles which we denote by $A_{\triangle}$. We obtain

$$\frac{|AB|}{|BC|} \times \frac{|DF|}{|FA|} = \frac{A_{\triangle ABF}}{A_{\triangle CBF}} \times \frac{A_{\triangle DBF}}{A_{\triangle ABF}}.$$

Further, notice that CE and ED are the bases of $\triangle CBE$ and $\triangle DBE$, and also $\triangle CFE$ and $\triangle DFE$. Thus,

$$\frac{|CE|}{|ED|} = \frac{A_{\triangle CBE}}{A_{\triangle DBE}} = \frac{A_{\triangle CFE}}{A_{\triangle DFE}}$$

but

$$\frac{A_{\triangle CBE}}{A_{\triangle DBE}} = \frac{A_{\triangle CBF} + A_{\triangle CFE}}{A_{\triangle DBF} + A_{\triangle DFE}},$$

implying that

$$\frac{|CE|}{|ED|} = \frac{A_{\triangle CBF}}{A_{\triangle DBF}}.$$

Putting it together, we get

$$\frac{|AB|}{|BC|} \times \frac{|CE|}{|ED|} \times \frac{|DF|}{|FA|} = \frac{A_{\triangle ABF}}{A_{\triangle CBF}} \times \frac{A_{\triangle CBF}}{A_{\triangle DBF}} \times \frac{A_{\triangle DBF}}{A_{\triangle ABF}} = 1. \quad \square$$

Menelaus' Theorem is a straightforward yet valuable proposition that can be used to establish various other results.

3 Harmonic Points

Definition 3.1. Suppose four points A, B, C, and D lie on the same straight line as shown in Figure 2 below. These four points are called *harmonic points* if D divides the segment AB internally in the same proportion as C divides the segment AB externally, that is, if the following proportion holds

$$\frac{|AD|}{|DB|} = \frac{|AC|}{|BC|}.$$

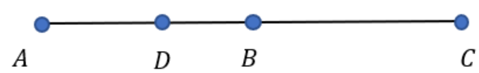

Figure 2: Harmonic points

Proposition 3.1. Refer to Figure 3 below. The points A, G, O, and F are harmonic points.

Proof: We use Menelaus' Theorem to prove this result. By this theorem, for $\triangle AOC$ and transversal line BGF, we have

$$\frac{|AG|}{|GO|} \times \frac{|OF|}{|FC|} \times \frac{|CB|}{|BA|} = 1.$$

Similarly, for $\triangle AOB$ and line HDC,

$$\frac{|AH|}{|HO|} \times \frac{|OD|}{|DB|} \times \frac{|CB|}{|CA|} = 1,$$

and for $\triangle COB$ and line AFD,

$$\frac{|OF|}{|FC|} \times \frac{|CA|}{|AB|} \times \frac{|BD|}{|OD|} = 1.$$

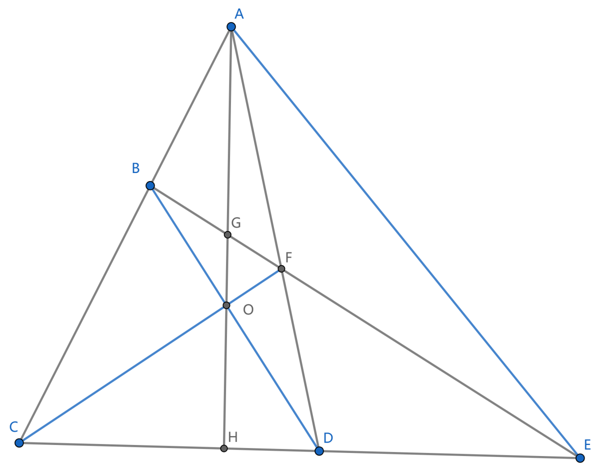

Figure 3: Harmonic points in a complete quadrilateral

Comparing the above equations above, we can easily see that $\dfrac{|AG|}{|GO|} = \dfrac{|AH|}{|HO|}$, which means that the points A, O, G, and H form a group of harmonic points. $\square$

Note: There are in fact many harmonic points. Can you identify some other instances of such points?

4 Miquel's Theorem and Miquel's Point

In 1838, a French mathematician Auguste Miquel (1816-1851) published the result in the *Journal de Mathématiques Pures et Appliquées* that stated the following theorem [2].

Theorem 4.2. Given a complete quadrilateral $ABCDEF$ (refer to Figure 4 below), consider the four triangles $\triangle CAD, \triangle BAF, \triangle CBE$, and $\triangle DFE$. Then the four circumcircles of these four triangles intersect at a single point M.

This statement is known as *Miquel's theorem* and the point M in question is referred to as *Miquel's point*.

Proof: We prove the theorem using angles in the argument. Let M be the intersection of circles BAF and DFE different from point F. Then it can be shown that $\angle CAM = \angle MFE = \angle MDE$, and hence the points C, A, M, and D lie on the same circle. Likewise, it can be demonstrated that points

C, B, M, and E lie on the same circle. Therefore, point M is the intersection of the four circles. □

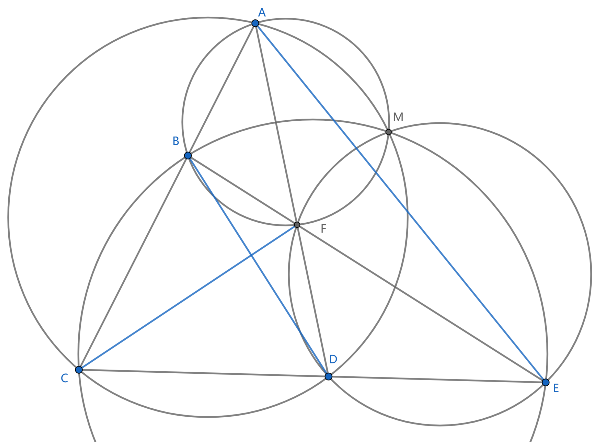

Figure 4: Miquel Point

5 Further Insights Into Miquel's Point

If additional restrictions are imposed on the complete quadrilateral $ABCDEF$, more relations involving Miquel's point may be derived. For example, suppose the vertices B, C, D, and F of a complete quadrilateral lie on the same circle (see Figure 5 below).

Upon inspecting the figure, we can draw the following potentially valid conclusions:

1. The points A, M, and E are collinear.

2. The points C, O, F, M, and B, O, D, M lie on the same circle.

3. Angles $\angle FMO$ and $\angle CMO$ are congruent, and angles $\angle BMO$ and $\angle DMO$ are congruent.

4. Segment MO is perpendicular to segment AE.

In fact, all of these statements hold true. We encourage you to attempt proving them independently as discovering these geometric insights firsthand is the most rewarding approach.

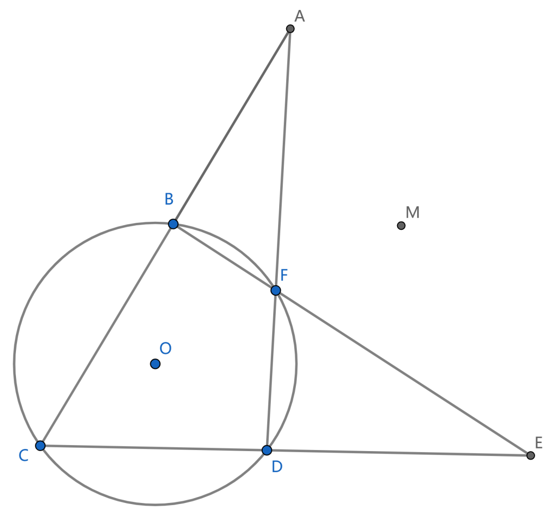

Figure 5: Quadrilateral $BCDF$ inscribed in a circle

6 Newton Line

Now we shift our sights onto the 17th century's Great Britain Island where another remarkable discovery was made by the renowned mathematician and physicist Isaac Newton. Here is his assertion [3].

Theorem 6.3. Consider a complete quadrilateral $ABCDEF$ as depicted in Figure 6 below. Suppose the points M, N, and L are the mid-points of the corresponding diagonals CF, BD, and AE. Then the three points M, N, and L lie on the same straight-line.

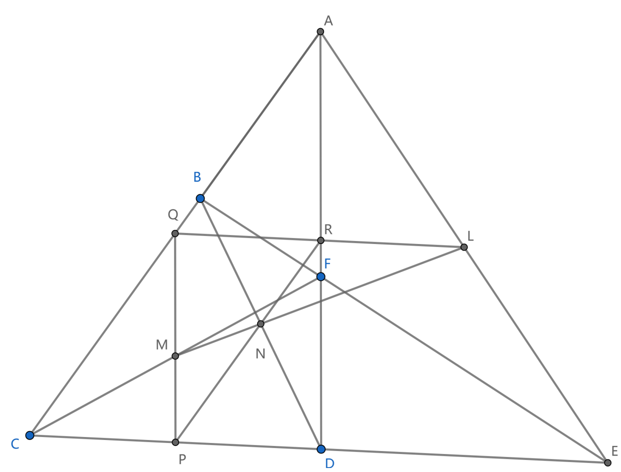

Figure 6: Newton Line

Proof: The line that contains the points M, N, and L is called the Newton line. Let P, Q, and R be the mid-points of the segments CD, AC, and AD, respectively. Then the points Q M, and P lie on the same line. Similarly, points P, N, and R, and also Q, R, and L line on corresponding straight-lines. Thus, we have

$$\frac{|QL|}{|LR|} = \frac{|CE|}{|ED|}, \frac{|QM|}{|MP|} = \frac{|AF|}{|FD|}, \quad \text{and} \quad \frac{|RN|}{|NP|} = \frac{|AB|}{|BC|}.$$

Now, using the Menelaus Theorem for the complete quadrilateral $ABCDEF$, we write

$$\frac{|QL|}{|LR|} \times \frac{|RN|}{|NP|} \times \frac{|PM|}{|QM|} = \frac{|CE|}{|ED|} \times \frac{|AF|}{|FD|} \times \frac{|AB|}{|BC|} = 1.$$

This shows that $LRQMPN$ is also a complete quadrilateral. Since it includes LMN as a side, the points L, M, and N are collinear. □

7 Gauss-Bodenmiller Theorem

To extend the observations further, consider the mid-points M, N, and L introduced in the previous section (see Figure 6 above). We know that they lie on Newton line. Consider also orthocenters (intersections of altitudes) of the four triangles $\triangle ABF$, $\triangle FDE$, $\triangle BCE$, and $\triangle ACD$, denoting them by H_1 through H_4, respectively (depicted in Figure 7 below).

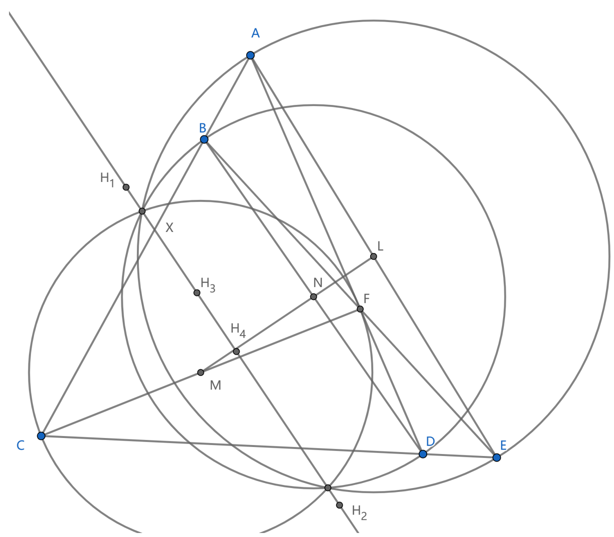

Figure 7: Illustration for Gauss-Bodenmiller theorem

The following is the Gauss-Bodenmiller Theorem that has been in existence for over 200 years [1],[3] (and was later rediscovered by the author of this article).

Theorem 7.4. The points H_1, H_2, H_3, and H_4 lie on the same line (called *Steiner line*) which is perpendicular to Newton line.

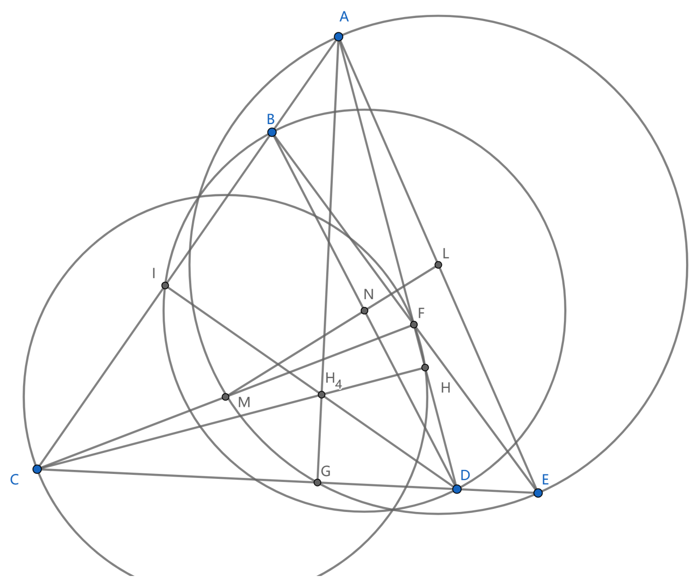

Figure 8: Orthocenter H_4 and feet G, H, and I of $\triangle ACD$

Proof: Consider, for instance, point H_4 which is the orthocenter of $\triangle ACD$. Let G, H, and I be the feet of the respective altitudes (see Figure 8 below). We know that the following proportions hold true for altitudes: $|HH_4| \times |H_4C| = |GH_4| \times |H_4A| = |IH_4| \times |H_4D|$.
Next, noticing that these three products give the powers (distances) of the point H_4 to the circles with centers M, L and N, and respective diameters CF, AE, and BD. This indicates that the circles have the same radical axis on which H_4 lies.

Finally, it can be shown similarly that the points H_1, H_2, and H_3 belong to the same radical axis. This proves the theorem. $\square$

8 Illustrative Example

Let $ABCDEF$ be a complete quadrilateral. Suppose the point G is the intersection of segments BD and CF (refer to Figure 9 below).

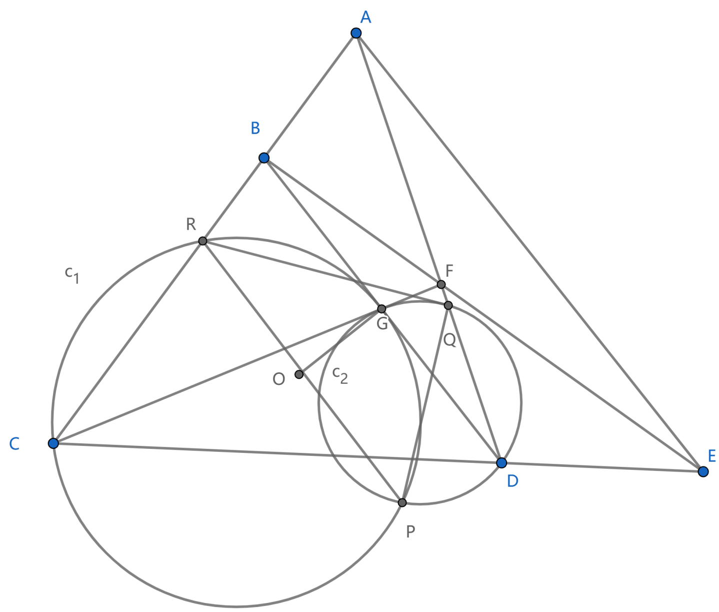

Figure 9: Illustrative Example

Let circle c_1 be tangent to BD and pass through the point C. Also, let circle c_2 be tangent to CF and pass through point D. Define the points R as the intersection of c_1 and AC and Q as the intersection of c_2 and AD. The circles c_1 and c_2 intersect at points G and P. Let the point O denote the circumcenter of $\triangle PQR$. We will show that the line OG is perpendicular to the line AE.

Proof: Refer to Figure 10 given on the next page. Let point S be the intersection of segments QG and AC, and let point T be the intersection of segments RG and AD. Let point L denote the intersection of segments ST and RQ.

Notice that the points C, R, G and P lie on the same circle, and segment CG is tangent to the circle c_2. Thus, we have that $\angle SRP = \angle CGP = \angle GQP = \angle SQP$, and so the points S, R, Q, and P are positioned on the same circle. Using symmetry, we can argue that the points T, R, Q, and P lie on the same circle as well. Therefore, all the five points P, Q, R, S, and T are located on the same circle, which is exactly the circumcircle of $\triangle PQR$ with the center O.

Consequently, in a complete quadrilateral $ARSTLQ$, the points R, S, T, and Q are situated on the same circle. By the Gauss-Bodenmiller Theorem, GO is perpendicular to AL.

Further, in complete quadrilaterals $ABCDEF$ and $ARSTLQ$, the sets of lines $\{AC, AD, AG, AE\}$, and $\{AR, AQ, AG, AL\}$ are harmonic lines. However, from the given information, AC and AR, AD and AQ, AG and AG are three pairs of the same lines, indicating that the fourth pair, AL and AE, must be the same line. Therefore, since we have already proven that GO is perpendicular to AL, we must have that GO is perpendicular to AE. $\square$

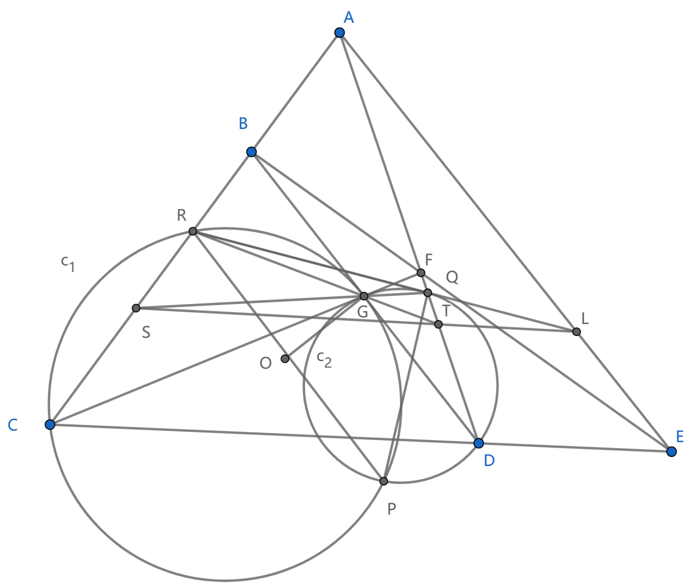

Figure 10: Solution for the Illustrative Example

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