Constructing All Convex Shapes Using a Modified Version of the Traditional Chinese Tangram Puzzle

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Abstract

It is well known that there are 13 convex tangram figures for the classical Chinese Tangram puzzle. In this paper, the focus shifts to a different variant where it is demonstrated that there are 11 such convex tangram figures. Furthermore, it is established that any convex tangram figure must have all of its border vertices situated on a single lattice. A computer program is developed to solve Diophantine equations and inequalities in order to identify candidate polygons. Subsequently, a manual construction process is employed to validate whether each candidate polygon qualifies as a legitimate tangram construction.

Keywords: classical Chinese tangram puzzle, tangram figures, convex, tangram variant, Diophantine equations

1 Introduction

1.1 Background

The Chinese Tangram puzzle is a traditional dissection puzzle consisting of a square that is divided into seven smaller geometric shapes, which are then reassembled to form various figures, such as animals, people, and objects. The puzzle is renowned for its versatility in stimulating creativity and problem-solving skills. Its origins trace back to China during the Song Dynasty (960-1279 AD), although some sources suggest it may have originated even earlier, possibly during the Tang Dynasty (618-907 AD) [1]. Initially known as the "seven boards of skill" or "seven clever pieces," it gained popularity in China as a recreational activity and educational tool.

The Tangram puzzle's introduction to Western audiences is credited to a few different sources. In the early 19th century, it was brought to Europe by trading ships, where it quickly captured the interest of puzzle enthusiasts. It gained further exposure through publications and exhibitions, becoming a popular pastime across Europe and North America by the mid-1800s. The puzzle's name, "Tangram," is believed to have been coined in the early 19th century, combining the words "Tang" (referring to the Tang Dynasty) and "gram" (from the Greek word for "something written" or "drawing"). It reflects the puzzle's presumed Chinese origins and its focus on geometric shapes.

Over time, the Tangram puzzle has remained a beloved classic, inspiring countless variations, artistic interpretations, and educational applications worldwide. Its enduring appeal lies in its simplicity, yet boundless potential for creativity and mental stimulation. The seven pieces of the classical Chinese tangram puzzle are depicted in Figure 1 below.

The seven tangram pieces typically referred to as "tans", can be described in terms the ratio of their side lengths. Without loss of generality we can let the smallest side length of all of the pieces be

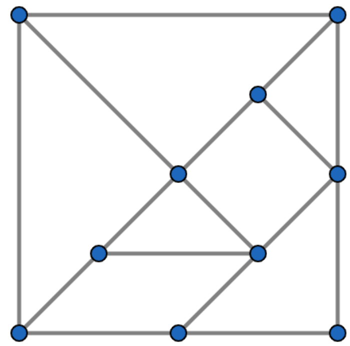

Figure 1: A classical Chinese tangram puzzle

equal to unity. Then the seven tans have the shapes described as follows:

- 2 isosceles right triangles with leg length 1,
- 1 isosceles right triangles with leg length $\sqrt{2}$,
- 2 isosceles right triangles with leg length 2,
- 1 parallelogram with side lengths of 1 and $\sqrt{2}$ and an angle of $\frac{\pi}{4}$,
- and 1 square with side length of 1.

In [2], it has been shown that there are exactly 13 distinct convex shapes that can be formed using all of the pieces above. Additionally, in [3] it has been proven for shapes that are not necessarily convex that there are one triangle, six quadrilaterals, 53 pentagons, and infinitely many n-gons for $n \geq 6$ using all of the tan pieces.

1.2 Modified Tangram Puzzle

This paper explores a variation of the tangram puzzle to identify all convex shapes achievable within this variant. The specific tangram variation under consideration is illustrated in Figure 2 below. Similarly, let the shortest side length of all of the pieces be equal to 1. Then the pieces of the tangram variation can be described as follows:

- 4 isosceles right triangles with leg length $\sqrt{2}$,
- 2 parallelograms of side lengths of 1 and $\sqrt{2}$ and an angle of $\frac{\pi}{4}$,
- and 1 square with a side length of $\sqrt{2}$.

In this paper, unless explicitly stated otherwise, all subsequent content pertains to this tangram variation and its individual tan pieces.

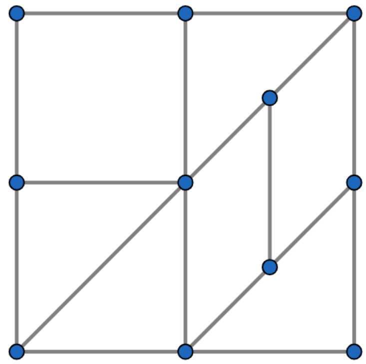

Figure 2: A modified tangram puzzle

2 Theoretical Framework

In this section, we establish theoretical foundations. Most of the definitions and theorems are similar to those in [4], which focuses on non-convex polygons. Therefore, we adapt the statements in [4] to suit the context of convex polygons in this paper. We start with the following definition.

Definition 2.1. A basic triangle *is a* 1×1 *isosceles right triangle.*

Then we have the following quick observations.

Lemma 2.1. All the tan pieces can be partitioned into non-overlapping basic triangles.

Note the partition of tan pieces might not be unique, and readers are free to choose partition patterns.

Definition 2.2. A generalized tangram *is a tangram can be partitioned into non-overlapping basic triangles.*

The convex shapes we seek are always generalized tangrams.

Definition 2.3. A lattice tangram *is a tangram where all of its border vertices can be placed onto a single lattice.*

The main theorem of this section is the following.

Theorem 2.1. *Let T be a generalized tangram. If T is convex, then T is a lattice tangram.*

We will prove this theorem through a series of lemmas and definitions below.

Lemma 2.2. *Given any two basic triangles R, S of a generalized tangram, let their lattices be U, V, respectively. Then U, V either have the same orientation or differ by an angle of $\frac{\pi}{4}$.*

Proof. Note that the lattices of any two adjacent tan pieces either have the same orientation or differ by an angle of $\frac{\pi}{4}$. Since a tangram have its tan pieces touching each other, the lattices of any two pieces of the tangram will have the same orientation or differ by an angle of $\frac{\pi}{4}$. □

By Lemma 2.2, all basic triangles of a generalized tangram can be partitioned into two sets P and Q such that lattices of basic triangles in P are translations of each other, lattices of basic triangles in Q are translations of each other, and lattices of basic triangles in Q are translations from rotation of lattices for basic triangles in P by $\frac{\pi}{4}$. We further partition the set P into $P_1, P_2, \cdots, P_n$ such that basic triangles in P_i share the same lattice where $1 < i < n$, and P_i and P_j have distinct lattices if $i \neq j$. We similarly partition Q into $Q_1, Q_2, \cdots, Q_m$. In the above, n, m are assumed to be some whole numbers. We summarize the above observation into the following lemma.

Lemma 2.3. *A generalized tangram T can be partitioned as $T = (P_1 \cup P_2 \cup ... \cup P_n) \cup (Q_1 \cup Q_2 \cup ... \cup Q_m)$ where $n, m \in \mathbb{W}$, where $P_1, P_2, ..., P_n, Q_1, Q_2, ..., Q_m$ are subsets of basic triangles. Basic triangles in the same subset (e.g. P_i or Q_j) have the same lattice. Basic triangles of P_{i_1} and P_{i_2} have their lattices differ by a translation where $i_1 \neq i_2$. Basic triangles of Q_{j_1} and Q_{j_2} have their lattices differ by a translation where $j_1 \neq j_2$. Basic triangles of P_i and Q_j differ by a translation after a rotation of $\frac{\pi}{4}$.*

Note each P_i and Q_j is potentially composed of multiple disjoint polygons.

Definition 2.4. *A point v of a polygon R of P_i (or Q_j) is consider a* vertex *if there is no circular disc D centered at v such that $D \cap R$ is a half-disc of D.*

Definition 2.5. *Boundary $bd(P_i)$ (or $bd(Q_j)$) is the union of the boundaries of all polygons with vertices defined in the definition 2.4 that compose P_i (or Q_j) .*

Lemma 2.4. *P_{i_1} and P_{i_2} (Or Q_{j_1} and Q_{j_2}) have no vertices in common if $i_1 \neq i_2$ ($j_1 \neq j_2$).*

Proof. Assume on the contrary, that there does exist a vertex belong to both P_{i_1} and P_{i_2}. Then, since they are of the same orientation, the must be the same lattice. We have arrived at a contradiction. □

Lemma 2.5. *P_i and Q_j where $1 \leq i \leq n, 1 \leq j \leq m$, share at most one common vertex.*

Proof. Assume on the contrary, P_i and Q_j share two vertices W, X. Without loss of generality, we can shift both coordinate systems to have the W as the origin. Then the coordinates of X in both coordinate system have integer components. Note the coordinate system of Q_j is a translation following a rotation of $\frac{\pi}{4}$ from the coordinate system of P_i. The rotation leads both coordinates X in Q_j to be irrational. Hence X is NOT on the lattice of Q_j after the rotation. The following translation operation can keep W or X on the lattice of Q_j but not both. This contradicts with the assumption that W and X are lattice points of Q_j.

□

Definition 2.6. *A point J is called a joint vertex of P_i and Q_j if J is a vertex of both P_i and Q_j.*

Definition 2.7. *A joint vertex L is called a V-vertex of P_i and Q_j if $L \subseteq bd(T)$.*

Lemma 2.6. *A generalized tangram T has a finite number of V-vertices.*

Definition 2.8. *A joint vertex K is called a T-vertex of P_i and Q_j with $P_{i'}$ (or $Q_{j'}$) above where $i \neq i'$ (or $j \neq j'$) if K is on the interior of a side of $P_{i'}$ (or $Q_{j'}$).*

Lemma 2.7. *A generalized tangram T has a finite number of T-vertices.*

Definition 2.9. *A directed segment $\overrightarrow{\mathrm{YZ}}$ is called a* primal segment *of P_i and Q_j if Y is a joint vertex of P_i and Q_j, and Z is a vertex of P_i or Q_j and the segment does not contain any other vertices of P_i or Q_j in its interior.*

Lemma 2.8. *If $\overrightarrow{\mathrm{YZ}}$ is a primal segment of P_i and Q_j, then Z must be a T-vertex of T.*

Proof. By Lemma 2.5, Z can be in one of P_i or Q_j but not both. Without loss of generality, we assume $Z \in P_i$. Then $Z \notin Q_j$ or Z is a relative interior point of Q_j. Hence $Z \notin bd(T)$ since T is convex. Since Z is in the interior of T, there must be another $P_{i'}$ or $Q_{j'}$ different from P_i and Q_j having Z as a vertex. By Lemma 2.4, the other piece must be $Q_{j'}$. Hence Z is a T-vertex. □

Now, we can begin our proof for theorem 2.1.

Proof. If $bd(T) \subseteq P$ or $bd(T) \subseteq Q$, without loss of generality, we assume $bd(T) \subseteq P$. Let $P_{i_1}, \cdots, P_{i_r}$ have non-empty intersection with $bd(T)$, and $bd(T) \subseteq P_{i_1} \cup \cdots \cup P_{i_r}, r \in \mathbb{N}$. If $r > 1$, since $bd(T)$ is a simple polygon, there exist $1 \leq s, t \leq r$ and $s \neq t$ such that P_{i_s} and P_{i_t} has a common vertex. This contradicts with the conclusion that P_{i_s} and P_{i_t} have no common vertex by Lemma 2.4. Hence, $r = 1$ and the $bd(T)$ belongs to one lattice, and the proof is complete.

Otherwise, there exists a V-vertex $A_0 \in bd(T)$ such that A_0 is a joint vertex of P_{i_0} and Q_{j_0}. Let $\overrightarrow{\mathrm{A_0A_1}}$ be a primal segment. By Lemma 2.8, A_1 is a T-vertex of T. Without loss of generality, we may assume P_{i_0} is on the left of $\overrightarrow{\mathrm{A_0A_1}}$, and Q_{j_0} is on the right of $\overrightarrow{\mathrm{A_0A_1}}$.

Following the same procedure, we can find another primal segment $\overrightarrow{\mathrm{A_1A_2}}$, By Lemma 2.8, A_2 is a T-vertex of T. We can maintain that a P_{i_1} is on the left of $\overrightarrow{\mathrm{A_1A_2}}$, and a Q_{j_1} is on the right of $\overrightarrow{\mathrm{A_1A_2}}$. This is because if $A_1 \in P_{i_0}$, the arc A_0, A_1, A_2 would turn left at A_1, maintaining the fact that P_{i_0} is on the left of $\overrightarrow{\mathrm{A_1A_2}}$, and a new Q_{j_1} is on the right of $\overrightarrow{\mathrm{A_1A_2}}$; and if $A_1 \in Q_{j_0}$, the arc A_0, A_1, A_2 would turn right at A_1, also maintaining the fact a new P_{i_1} is on the left of $\overrightarrow{\mathrm{A_1A_2}}$, and Q_{j_0} is on the right of $\overrightarrow{\mathrm{A_1A_2}}$.

We can recursively define $A_3, A_4, \ldots$ maintaining the fact that a P_i region is to the left of the new primal segment and a Q_j region is to the right of the new primal segment. By Lemma 2.7, the $A_0, A_1, \cdots$ sequence must loop back. Note A_0 is not a T-vertex. There exist $b, c \in \mathbb{N}, b < c$, such that $A_b = A_c$. Without loss of generality, we can assume that b is the smallest such a natural number.

By Lemma 2.4, the vertex A_b (and A_c) is a joint vertex of exactly one P_i piece and one Q_j piece. Hence the two primal segments coming into A_b (and A_c) must be the same. Hence $A_{b-1} = A_{c-1}$. This contradicts with the minimality of b. This contradiction proves that $bd(T) \subseteq P$ or $bd(T) \subseteq Q$, and the proof is complete.

□

3 Algorithm for Identifying Candidate Polygons

The derivation of algorithms in this section is based on [?].

Lemma 3.9. *T must have an area of 8.*

The lemma can be trivially verified by summing the areas of all of the pieces.

3.1 Maximum Polygon Size

All angles of a convex tangram are in the form $\frac{k\pi}{4}$, where $k \in \{1, 2, 3\}$. Let, n be the number of sides of a convex tangram, and s, m, l be the number of angles of size $\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$, respectively. Now, we have the following equations.

$$\begin{cases} s + m + l = n \\ s + 2m + 3l = 4(n-2) \end{cases}$$

Which is simplified to

$$\begin{cases} s + m + l = n \\ 3s + 2m + l = 8. \end{cases}$$

Hence,
$n = s + m + l \leq 3s + 2m + l = 8.$

3.2 Side Lengths and Candidate Solutions

Since angles of a convex tangram are always in the form of $\frac{\pi}{4}$, $\frac{2\pi}{4}$ or $\frac{3\pi}{4}$. The sides of the boundary polygon always rotates from its previous side by multiple of $\frac{\pi}{4}$. Hence the tangram can always be oriented in a rectangle as illustrated in Figure 3 that follows.

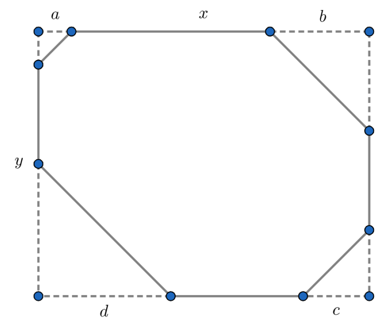

Figure 3: Orientation of tangram in a rectangle

Let x, y be the length and width of the rectangle, respectively. Without loss of generality, we may assume that $x \geq y$. Let, a, b, c, d be the side lengths of the corners removed in order to produce the original tangram shape, in the same order as the diagram above. Again, without loss of generality, we may assume that $a \geq b, c, d$.

By Theorem 2.1, any generalized tangram can be placed in a lattice. Hence a, b, c, d, x, y are all whole numbers.

Since the corners are non-overlapping, we have the following.

$$\begin{cases} a + b \leq x \\ c + d \leq x \\ a + d \leq y \\ b + c \leq y \end{cases}$$

By Lemma 3.9, we also have the following.

$2xy - (a^2 + b^2 + c^2 + d^2) = 16$

Since x, y are integers, they are less or equal to the sum of the maximum rational side lengths of all pieces. Hence

$x, y \leq 4 \times 2 + 2 \times 1 = 10.$

We can now enumerate all possible combinations of x, y, a, b, c, d using the following python code.

```
for x in range(1, 11) :
    for y in range(1, x + 1) :
        for a in range(x + 1) :
            for b in range(min(a + 1, x - a + 1)) :
                for c in range(min(a + 1, x + 1)) :
                    for d in range(min(a + 1, x - c + 1)) :
                        if ((a + d <= y) and (b + c <= y) and
                              (2 * x * y - (a * *2 + b * *2+
                                    c * *2 + d * *2) == 16)) :
                          print((x, y, a, b, c, d))
```

The following are the computed results.

$(3, 3, 1, 0, 0, 1), (3, 3, 1, 0, 1, 0), (3, 3, 1, 1, 0, 0),$
$(4, 2, 0, 0, 0, 0), (4, 3, 2, 0, 2, 0), (4, 3, 2, 2, 0, 0),$

$(4,4,2,2,2,2),(4,4,4,0,0,0),(5,2,1,1,1,1),$
$(5,2,2,0,0,0),(5,3,3,1,2,0),(5,3,3,2,1,0),$
$(5,5,4,1,4,1),(5,5,5,0,3,0),(6,2,2,0,2,0),$
$(6,2,2,2,0,0),(6,4,4,0,4,0),(8,1,0,0,0,0),$
$(9,1,1,0,1,0),(9,1,1,1,0,0),(9,8,8,0,8,0)$

The only case we will have to ignore due to symmetry is $(3,3,1,1,0,0)$ which is the same as $(3,3,1,0,0,1)$.

4 Identification of Valid Convex Tangrams

In this section, we go through all the candidate solutions from the previous section, and determine which can produce valid tangrams.

Theorem 4.2. *There are 11 convex tangrams that can be constructed from the tan variant in section 1.2.*

The theorem is proved in the following sub-sections.

4.1 Solutions Without Valid Tangrams

The following solutions do not have valid convex tangrams.

$(3,3,1,0,0,1),(4,2,0,0,0,0),(4,3,2,0,2,0),$
$(4,3,2,2,0,0),(5,3,3,1,2,0),(8,1,0,0,0,0),$
$(9,1,1,0,1,0),(9,1,1,1,0,0),(9,8,8,0,8,0)$

We prove the non-existence of tangrams for the above solutions one-by-one.

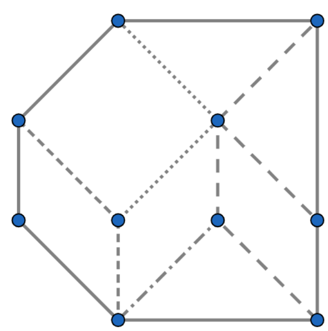

Figure 4: A tangram with side lengths $(3,3,1,0,0,1)$

In order to have a length of 3 on the right, it must be composed of a parallelogram and an isosceles right triangle. Without loss of generality, it may be placed as shown by the long dashed lines in the diagram.

In order to produce the side length of 1 on the left, it must be composed of a single parallelogram. Without loss of generality, it may be placed as shown by the medium dashed lines in the diagram.

There is now only one place to the square, shown by the short dashed lines in the diagram.

There is also only one way to place the remaining isosceles right triangles, shown by the dotted dashed lines in the diagram.

We can now see that we will have one isosceles right triangle that would not be able to fit into the shape, making this impossible to tile.

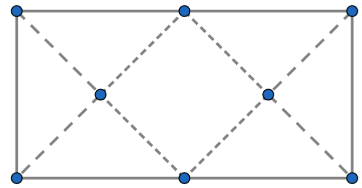

Figure 5: A tangram with side lengths $(4, 2, 0, 0, 0, 0)$

Notice that the two parallelograms cannot fit next to each other on the vertical sides. Therefore, they must be filled with the isosceles right triangles. They are shown by the long dashed lines in the diagram.

There is now only one place the square, as shown by the medium dashed lines in the diagram.

We can now see that the two parallelograms would not be able to fit into the shape, making this impossible to tile.

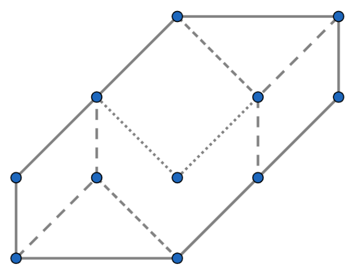

Figure 6: A tangram with side lengths $(4, 3, 2, 0, 2, 0)$

In order to have a length of 1 on the bottom left and top right, the parallelograms must be placed as shown in the diagram by the long dashed longs.

In order to have a length of 2 on the bottom and the top, we must place two isosceles right triangles as shown in the diagram by medium dashed lines.

Now, the square must be placed to be either touching the top or the bottom. Without loss of generality, it may be placed on the top in the diagram as shown by the short dashed lines.

We can now see that the two remaining isosceles right triangles would not be able to fit into the shape, making this impossible to tile.

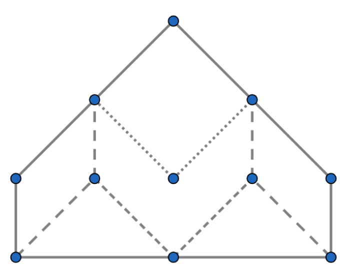

Figure 7: A tangram with side lengths $(4, 3, 2, 2, 0, 0)$

In order to have a length of 1 on the bottom left and bottom right, the parallelograms must be placed as shown in the diagram by the long dashed longs.

In order to have a length of 4 on the bottom, we must place two isosceles right triangles as shown in the diagram by medium dashed lines.

Now, the square must be placed to be either touching the top or the bottom. Without loss of generality, it may be placed on the top in the diagram as shown by the short dashed lines.

We can now see that the two remaining isosceles right triangles would not be able to fit into the shape, making this impossible to tile.

One can also use a similar set of arguments to prove that $(5, 3, 3, 1, 2, 0)$ is impossible to tile, though it requires more case work.

All of $(8, 1, 0, 0, 0, 0), (9, 1, 1, 0, 1, 0), (9, 1, 1, 1, 0, 0)$, $(9, 8, 8, 0, 8, 0)$ cannot be tiled due to having width ≤ 1, which makes the square unable to be fitted in, making all of them impossible to tile.

4.2 Valid Tangrams and Their Dissections

Eliminating all the impossible case, we arrive at the list of the ones that can produce a valid convex tangram shape. The are 11 such shapes: $(3, 3, 1, 0, 1, 0)$, $(4, 4, 2, 2, 2, 2)$, $(4, 4, 4, 0, 0, 0)$, $(5, 2, 1, 1, 1, 1)$, $(5, 2, 2, 0, 0, 0)$, $(5, 3, 3, 2, 1, 0)$, $(5, 5, 4, 1, 4, 1)$, $(5, 5, 5, 0, 3, 0)$, $(6, 2, 2, 0, 2, 0)$, $(6, 2, 2, 2, 0, 0)$, and $(6, 4, 4, 0, 4, 0)$. Below are specific examples of constructions (see Figures 8-18). It's important to note that certain convex tangrams can be constructed in multiple ways under geometric isometry. We have identified all convex shapes achievable with this variant of the tangram puzzle, thereby concluding this article.

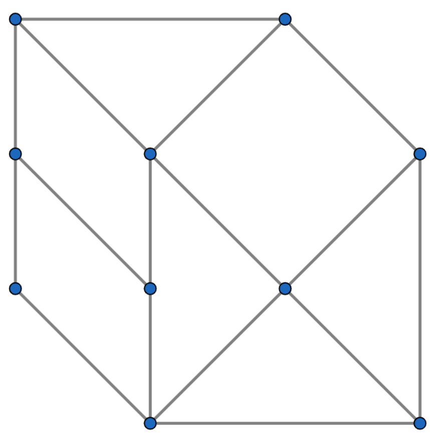

Figure 8: A tangram with side lengths $(3, 3, 1, 0, 1, 0)$

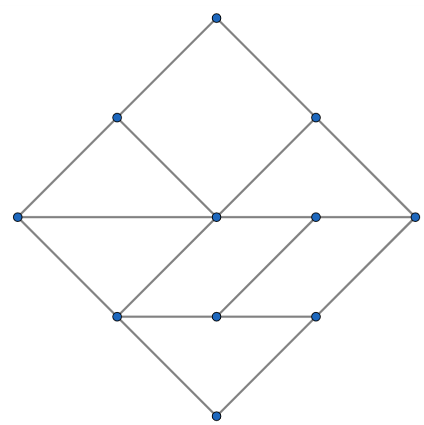

Figure 9: A tangram with side lengths $(4, 4, 2, 2, 2, 2)$

5 Future Work

In the future, we intent to replace the manual construction of tangrams in section 4 with a program algorithm. The tangram variant pieces can be generalized to any number of simple polygons that can be constructed from basic triangles. Lastly, we can also derive general results for non-convex tangrams as did by Sarah Sophie Pohl and Christian Richter [3].

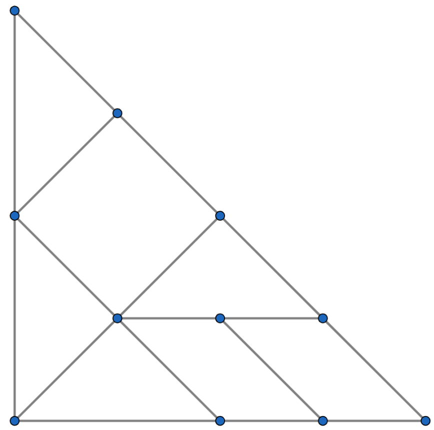

Figure 10: A tangram with side lengths $(4, 4, 4, 0, 0, 0)$

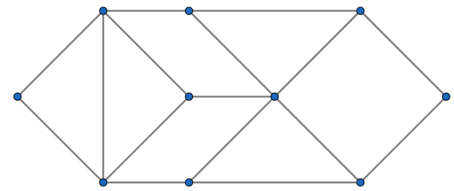

Figure 11: A tangram with side lengths $(5, 2, 1, 1, 1, 1)$

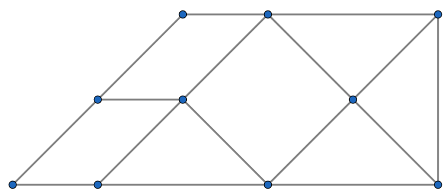

Figure 12: A tangram with side lengths $(5, 2, 2, 0, 0, 0)$

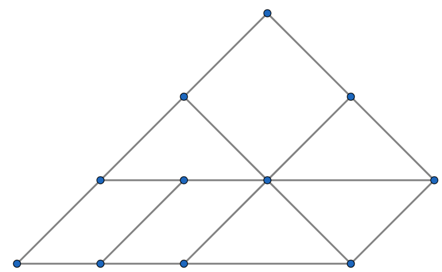

Figure 13: A tangram with side lengths $(5, 3, 3, 2, 1, 0)$

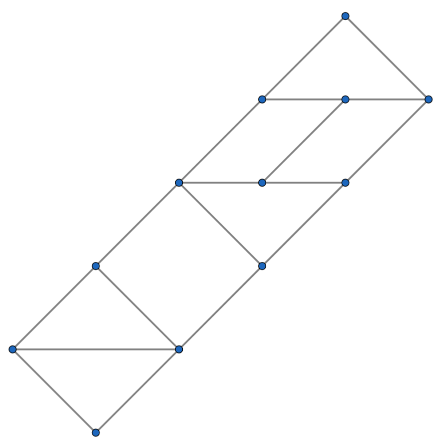

Figure 14: A tangram with side lengths $(5, 5, 4, 1, 4, 1)$

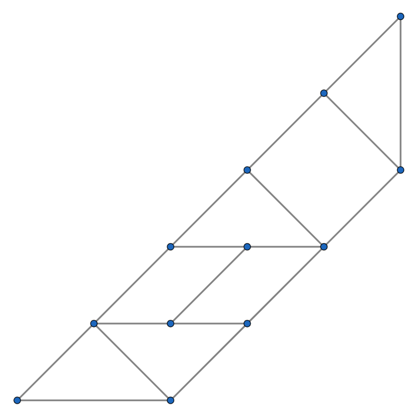

Figure 15: A tangram with side lengths $(5, 5, 5, 0, 3, 0)$

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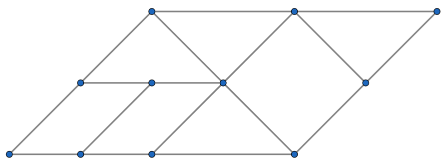

Figure 16: A tangram with side lengths $(6, 2, 2, 0, 2, 0)$

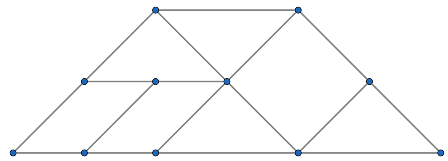

Figure 17: A tangram with side lengths $(6, 2, 2, 2, 0, 0)$

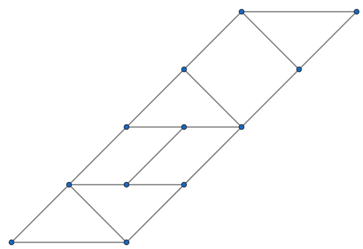

Figure 18: A tangram with side lengths $(6, 4, 4, 0, 4, 0)$

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[4] Scott, P. 2006. Convex Tangrams. *The Australian Mathematics Teacher*, 62(2), 2-5.