# The Story of Perfect Numbers from Euclid to Modern Computing

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**Abstract**

This essay explores the connection between perfect numbers and Mersenne primes, going over history and key properties with some proofs. It also includes Python code to generate the first eight perfect numbers and concludes with a brief overview of other numbers with interesting characteristics.

**Keywords**: Perfect number, Mersenne prime, Great Internet Mersenne Prime Search (GIMPS), proper divisor, abundant number, deficient number, triangular number, multi-perfect number, amicable pair, hyperperfect number

## 1 Introduction

A **perfect number** is a positive integer whose proper divisors (all positive divisors excluding the number itself) sum exactly to the number itself. For example, 6 is a perfect number. Its proper divisors are 1, 2, and 3, and their sum is $1 + 2 + 3 = 6$.

The deep connection between perfect numbers and **Mersenne primes** originates from a result first recorded by Euclid around 300 BC [1] and later generalized by the mathematician Leonhard Euler (1707 – 1783).[2] A Mersenne prime is a prime number of the form $2^p - 1$ where $p$ itself is a prime number. They are named after Marin Mersenne (1588 - 1648), a French polymath who studied these special prime numbers. Euclid proved that if $2^p - 1$ is prime, then the number $2^{p-1}(2^p - 1)$ is a perfect number.[3] This formula provides a direct method for constructing perfect numbers using Mersenne primes.

Over two thousand years later, Euler showed that every even perfect number must be of this form. In other words, there is a one-to-one correspondence between even perfect numbers and Mersenne primes. This elegant relationship narrows the search for perfect numbers to the search for Mersenne primes. As of today, all known perfect numbers are even, and they have been found by identifying Mersenne primes. The largest known perfect numbers correspond to the largest known Mersenne primes, many of which have been discovered through distributed computing projects such as the **Great Internet Mersenne Prime Search (GIMPS)** that started in 1996.[4] Its official site is *https://www.mersenne.org*.

## 2 Historical Perspective

Perfect numbers were first explicitly described in about 300 BC by Euclid in *Elements, Book IX*.[1] The earliest known perfect numbers are 6, 28, 496, and 8128. Throughout antiquity, evidence suggests that various civilizations, including the Egyptians and Romans, were intrigued by perfect numbers, and the Greeks often attributed mystical or magical properties to them. [1]

Later, Nicomachus of Gerasa (circa 100 AD), a significant figure in the Pythagorean school, provided one of the earliest classifications of numbers based on their divisors, known then as *aliquot parts* (proper divisors). In his text *Introduction to Arithmetic* [5], Nicomachus classified numbers into three categories:

- **Abundant numbers**: the sum of aliquot parts exceeds the number itself,

- **Deficient numbers**: the sum of aliquot parts is less than the number itself,

and

- **Perfect numbers**: the sum of aliquot parts equals the number itself.

These classifications laid the foundational concepts that influenced number theory for centuries.

## 3 Euclid-Euler Theorem with Proof

**Theorem.** An even integer $N$ is a perfect number if and only if $N = 2^{p-1}(2^p - 1)$ where $p$ and $2^p - 1$ are both prime numbers. That is, an even number is perfect if and only if it is of the form $2^{p-1}(2^p - 1)$ where $2^p - 1$ is a Mersenne prime.

For instance, the smallest perfect numbers 6, 28, 496, and 8128 can be written in the form $6 = 2 \cdot 3 = 2^{2-1}(2^2-1)$, $28 = 4 \cdot 7 = 2^{3-1}(2^3-1)$, $496 = 16 \cdot 31 = 2^{5-1}(2^5-1)$, and $8128 = 64 \cdot 127 = 2^{7-1}(2^7-1)$. Note that 2, 3, 5, and 7 are prime numbers, and $2^2-1 = 3, 2^3-1 = 7, 2^5-1 = 31$, and $2^7-1 = 127$ are Mersenne primes.

**Proof:** (See [6]). First, we will show that if $p$ and $2^p - 1$ are primes, then $= 2^{p-1}(2^p-1)$ is perfect. This is Euclid's part of the theorem, from *Elements, Book IX.*

Let $M = 2^p - 1$ denote a Mersenne prime. Then $N = 2^{p-1} \cdot M$. The divisors of $2^{p-1}$ are $1, 2, 4, 8, \cdots, 2^{p-1}$, and those of $M$ are 1 and $M$. We compute the sum of divisors of $N$, denoted by $\sigma(N)$. Since $N = 2^{p-1} \cdot M$ and $gcd(2^{p-1}, M) = 1$, we have

$$\sigma(N) = \sigma(2^{p-1}) \cdot \sigma(M).$$

Now,

$$\sigma(2^{p-1}) = 1 + 2 + 4 + 8 + \cdots + 2^{p-1} = \frac{2^p - 1}{2 - 1} = 2^p - 1 = M,$$

and

$$\sigma(M) = 1 + M.$$

Thus,

$$\sigma(N) = M \cdot (1 + M) = (2^p - 1)(1 + 2^p - 1) = 2^p(2^p - 1) = 2N.$$

Since the sum of all divisors of $N$ is $2N$, the sum of proper divisors is $2N - N = N$, and so, $N$ is perfect by definition.

Next, we want to show that if $N$ is an even perfect number, then it is of the form $2^{p-1}(2^p - 1)$ where $2^p - 1$ is a prime. This is what Leonhard Euler proved in the 18th century.

Let $N$ be an even perfect number. Then $N = 2^k\, m$ where $m$ is odd and $k \geq 1$. Since $N$ is perfect, $\sigma(N) = 2N = 2^{k+1}\, m$. Also, since $gcd(2^k, m) = 1$,

$$\sigma(N) = 2N = \sigma(2^k) \cdot \sigma(m) = (2^{k+1} - 1) \cdot \sigma(m).$$

Hence,

$$(2^{k+1} - 1) \cdot \sigma(m) = \sigma(N) = 2N = 2^{k+1}\, m,$$

or

$$\sigma(m) = \frac{2^{k+1}\, m}{2^{k+1} - 1} = m + \frac{m}{2^{k+1} - 1}.$$

Further, since $\sigma(m)$ is an integer, $2^{k+1}-1$ must be a divisor of $m$. Let $m = (2^{k+1}-1)\cdot q$ for some integer $q$. Substituting back into $N$, we obtain

$$N = 2^k(2^{k+1}-1)\cdot q.$$

From here, assuming $2^{k+1}-1$ is a prime number,

$$2N = \sigma(N) = \sigma(2^k)\cdot\sigma(2^{k+1}-1)\cdot\sigma(q) = (2^{k+1}-1)\cdot 2^{k+1}\cdot\sigma(q) = 2N\cdot\sigma(q),$$

and so, $\sigma(q) = 1$, or equivalently, $q = 1$. Taking $p = k+1$, we see that

$$N = 2^{p-1}(2^p-1)$$

where $2^p - 1$ is a prime.

## 4 Open Questions

Despite centuries of study, many fundamental questions about the existence, structure, and distribution of perfect numbers remain unresolved. Here are some of the fundamental open questions ([6],[7]):

- **Are there infinitely many even perfect numbers?** All known perfect numbers are even, and they correspond to Mersenne primes. Whether there are infinitely many Mersenne primes is still unknown.[8]
- **Do any odd perfect numbers exist?** No odd perfect number has ever been found. [8]
- **If odd perfect numbers exist, are there finitely or infinitely many of them?** Beyond existence, it's unknown whether there could be finitely or infinitely many odd perfect numbers.[9]
- **How are perfect numbers distributed?** There is no clear understanding of how sparse or dense perfect numbers are along the number line (though empirically, they become extremely rare).[10]

## 5 List of Perfect Numbers

The Python code in Figure 1 below outputs a list of the first eight perfect numbers, using a formula based on prime numbers. It applies the equation $2^{p-1}(2^p-1)$ where $p$ is a prime up to 31. For each prime $p$, the code checks if $2^p-1$ is also prime. If it is, the corresponding perfect number is calculated and returned. The code continues this process with the next prime value of $p$ until it reaches 31.

Python Code and Output

```
def factorization(num):
    fact_sum = 0
    for i in range(1, num + 1):
        if num % i == 0:
            fact_sum += i
    return fact_sum

perfect = []
usedprimes = []

primes = [2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31]
for p in primes:
    fact_sum = factorization(2**p - 1)
    if fact_sum == 2**p:
        perfect.append(2**(p - 1) * (2**p - 1))
        usedprimes.append(p)

print(perfect)
print(usedprimes)
```

```
[6, 28, 496, 8128, 33550336, 8589869056, 137438691328, 2305843008139952128]
[2, 3, 5, 7, 13, 17, 19, 31]
```

**Figure 1.** Python code with generated list of eight perfect numbers and corresponding values of $p$.

Today, through the GIMPS project, 52 Mersenne primes are known, and thus, 52 perfect numbers are known. They correspond to the prime numbers $p$ p = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243, 110503, 132049, 216091, 756839, 859433, 1257787, 1398269, 2976221, 3021377, 6972593, 13466917, 20996011, 24036583, 25964951, 30402457, 32582657, 37156667, 42643801, 43112609, 57885161, 74207281, 77232917, 82589933, and 136279841.

# 6 Curious Properties of Perfect Numbers

Perfect numbers exhibit a remarkable set of mathematical properties that have intrigued mathematicians for centuries. Below, we list a few of the most elegant ones. It is no wonder that, through the centuries, scholars have linked them to philosophical and theological ideas of "perfection". [6]

- Every even perfect number ends either in 6 or 8. No proof will be given here. For example, the first few perfect numbers that we generated in the previous section are: 6, 28, 496, 8128, 33550336,

$8589869056$, and $137438691328$.[11]

• Every even perfect number $N = 2^{p-1}(2^p - 1)$ is **triangular**, that is, $N$ can be written as the sum of consecutive integers from 1 to $2^p - 1$. This is easy to show. Indeed,

$$1 + 2 + 3 + \cdots + 2^p - 1 = \frac{2^p\,(2^p-1)}{2} = 2^{p-1}(2^p - 1) = N.$$

For example, $6 = 1 + 2 + 3$, $28 = 1 + 2 + 3 + 4 + 5 + 6 + 7$, and $496 = 1 + 2 + \cdots + 31$.

• Every even perfect number $N = 2^{p-1}(2^p - 1)$ has a binary representation consisting of $p$ ones followed by $p - 1$ zeros. This fact is a direct consequence of the expression

$$2^{p-1}(2^p - 1) = 2^{p-1}(1 + 2 + 4 + \cdots + 2^{p-1}) = 2^{p-1} + 2^p + 2^{p+1} + \cdots + 2^{2p-2}$$
$$= 0 \cdot 2^0 + 0 \cdot 2^1 + \cdots + 0 \cdot 2^{p-2} + 1 \cdot 2^{p-1} + 1 \cdot 2^p + \cdots + 1 \cdot 2^{2p-2} = \underbrace{1\,1 \ldots 1}_{p \text{ digits}}\;\underbrace{0\,0 \ldots 0_2}_{p-1 \text{ digits}}.$$

For example, $6_{10} = 100_2$, $28_{10} = 11100_2$, and $496_{10} = 111110000_2$.

• For every even perfect number, the reciprocals of all its divisors sum up to 2. Let $d_1, \ldots, d_k$ denote all proper divisors of $N$, we write

$$\frac{1}{N} + \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_k} = \frac{\sum_{i=1}^{k} d_i + N}{N} = \frac{2N}{N} = 2.$$

For example,

$$\frac{1}{6} + \frac{1}{3} + \frac{1}{2} + \frac{1}{1} = \frac{1+2+3+6}{6} = \frac{6+6}{6} = 2,$$
$$\frac{1}{28} + \frac{1}{14} + \frac{1}{7} + \frac{1}{4} + \frac{1}{2} + \frac{1}{1} = \frac{1+2+4+7+14+28}{28} = \frac{28+28}{28} = 2,$$

and

$$\frac{1}{496} + \frac{1}{248} + \frac{1}{124} + \frac{1}{62} + \frac{1}{31} + \frac{1}{16} + \frac{1}{8} + \frac{1}{4} + \frac{1}{2} + \frac{1}{1} = \frac{1+\cdots+248+496}{496} = \frac{496+496}{496} = 2.$$

# 7 Generalizations of Perfect Numbers

In number theory, various generalizations of perfect numbers, such as multi-perfect numbers, amicable numbers, and hyper-perfect numbers, have fascinated mathematicians for centuries due to their rich algebraic structure and rarity. Below, we explore each of these classes in more detail.

### 7.1 Multi-Perfect Numbers

A **multi-perfect number** (or **$k$-perfect number**) is a natural number $N$ such that the sum of its divisors (including $N$ itself) equals $k \cdot N$ for some integer $k > 1$. For example, 6 is a 2-perfect number, since $1 + 2 + 3 + 6 = 12 = 2 \cdot 6$, so $k = 2$. Similarly, $120 = 2^3 \cdot 3 \cdot 5$ is a 3-perfect number because its divisors are $1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60$, and 120, which add up to $1 + 2 + \cdots + 120 = 360 = 3 \cdot 120$, and so, $k = 3$. It can also be shown that $30240 = 2^5 \cdot 3^3 \cdot 5 \cdot 7$ is the smallest 4-perfect number.[11]

### 7.2 Amicable Pairs

Two numbers, $m$ and $n$, are called an **amicable pair** if each is the sum of the proper divisors of the other. Formally, $\sigma(m) = n$ and $\sigma(n) = m$. For example, $(220, 284)$ is the smallest amicable pair. The sum of the proper divisors of $220 = 2^2 \cdot 5 \cdot 11$ is $\sigma(220) = 1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284$, while the sum of the proper divisors of $284 = 2^2 \cdot 71$ is $\sigma(284) = 1 + 2 + 4 + 71 + 142 = 220$.[12]

The first ten amicable pairs are $(220, 284)$, $(1184, 1210)$, $(2620, 2924)$, $(5020, 5564)$, $(6232, 6368)$, $(10744, 10856)$, $(12285, 14595)$, $(17296, 18416)$, $(63020, 76084)$, and $(66928, 66992)$.[12]

### 7.3 Hyperperfect Numbers

A$k$**-hyperperfect number** is a natural number $N$ that satisfies the equation $N - 1 = k(\sigma(N) - 1)$. For example, $21 = 3 \cdot 7$ is a hyperperfect number with $k = 2$ since $\sigma(21) = 1 + 3 + 7 = 11$, and so, $N - 1 = 21 - 1 = 20$ and $\sigma(N) - 1 = \sigma(21) - 1 = 11 - 1 = 10$, giving $k = 2$.[13]

The list of smallest hyperperfect numbers for $k = 1, 2, 3, 4, 6, 10, 11$, and 12 is: 6, 21, 325, 1950625, 301, 159841, 10693, and 697.[13]


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