Symmetries of Frieze Patterns

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Abstract

Ornamental patterns have decorated art and architecture across cultures for millennia, long before their mathematical properties were formally understood. This paper explores the symmetry groups underlying ornamental and frieze patterns. We first examine one-dimensional ornamental patterns, governed by translations and reflections, and show that only two distinct symmetry groups are possible. Extending to two dimensions, we classify frieze patterns – infinitely repeating designs in one direction – into seven distinct groups based on combinations of translations, reflections, rotations, and glide reflections.

Keywords: ornamental pattern, frieze pattern, group, symmetry, transformation, translation, reflection, rotation, glide reflection

1 Introduction

Ornamental patterns have been a prominent feature in art, architecture, textiles, and everyday objects across nearly every culture and most of history. Ancient civilizations such as the Egyptians, Greeks, Romans, and Mesopotamians used linear decorative motifs to adorn pottery, clothing, and even temples. These early ornamental patterns often exhibited simple forms of repetition and symmetry, long before their mathematical properties were formally studied. In Islamic art (Figure 1), particularly from the 8th century onward, artisans developed repeating patterns, including friezes to adorn mosques and other religious buildings [1].

Figure 1: Fragment from a frieze with a meandering pattern and diamond-shaped rosettes, 6th century Islamic art, Egypt. Metropolitan Museum of Art, Accession Number 09.217.1a, b.

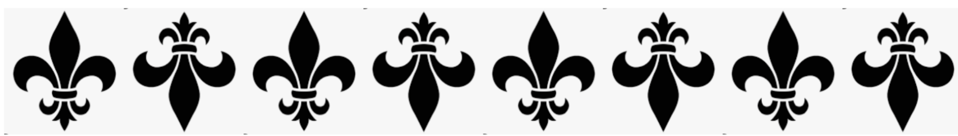

Figure 2: Example of a frieze pattern based on a fleur-de-lis image generated using the Python code developed in this paper (Supplementary Information).

Frieze patterns (Figure 2), as a formal mathematical concept, emerged much later through the study of symmetry in the 19th and 20th centuries. Mathematicians began to classify all possible one-dimensional repeating patterns based on their symmetries, eventually identifying exactly seven distinct frieze groups. However, this work was rarely focused exclusively on frieze or ornamental patterns. Instead, writing by authors such as Fedorov in 1891 [2] and Polya in 1924 [3] focused on wallpaper patterns (patterns that exist in two dimensions and have infinite rapport in both directions — both Polya and Fedorov proved that there are only 17 groups that include all wallpaper patterns independently from each other) or on similar patterns in three dimensions. These more formal classifications of wallpaper patterns made it easier for artists such as M. C. Escher to design works of art centering on the symmetries of wallpaper patterns [4].

This expository essay explores the mathematical classification of frieze patterns and ornamental patterns. We can understand frieze patterns better by first studying a simpler version, the **line** or **ornamental pattern**, a pattern that exists in only one dimension and can be more easily imagined as a colored string. For illustrative purposes, different

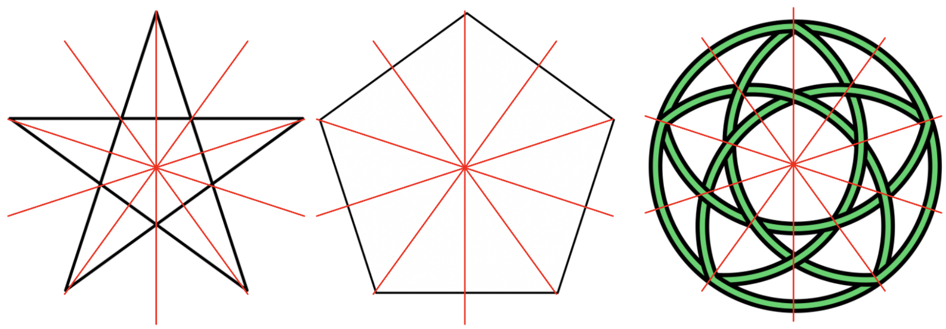

Figure 3: Three objects (pentagram, pentagon, and Celtic knot) each with fivefold reflective and rotational symmetry. The lines of reflection are marked with red lines.

colors on a string can be better represented by beads of different sizes, allowing us to make use of size to distinguish between different parts of the pattern. Ornamental patterns only consist of translation and reflection symmetries. As mentioned above, rotation in one dimension is impossible. Glide reflection is also impossible in one dimension, as it would involve reflection over the line of translation, which is impossible in one dimension.

1.1 Key Concepts

A **pattern** is a geometric design that exists in one or more dimensions and repeats in one or more directions, creating a sense of visual consistency and structure. **Frieze patterns** are endlessly repeating designs (known as patterns of **Infinite Rapport**) that exist in the two-dimensional plane. They can be imagined as a strip that stretches infinitely but has a finite width. **Translation** is a transformation that moves every point of a shape the same distance in the same direction, without rotating or flipping it. The space remains congruent and oriented the same way. **Reflection** is a transformation that flips a shape over a specific line, called the line of reflection, producing a mirror image of the original shape. **Rotation** is a transformation that turns a shape around a fixed point, known as the center of rotation, by a certain angle and in a specified direction (clockwise or counterclockwise). Notably, it is only possible in a space of two or more dimensions, since in one-dimensional space, only rotation by 180 degrees is possible, with the result being the same as reflection. **Glide reflection** is a combination of a translation and a reflection. First, the shape being transformed is translated along a line and then reflected over that same line or a line parallel to it [5]. These transformations are all types of **isometries**, meaning they preserve distances and angles, keeping the shape congruent to its original form. In fact, these transformations form a **symmetry group** which we define rigorously below.

A **transformation** is a mapping of space that associates with every point p in the space a point p'. Transformations are written as capital letters, such as transformation S, transformation T^{-1} and so on [6].

A special transformation carries a point p on itself (the points p and p' occupy the same coordinates). This transformation is known as the **identity** of the space and is marked with the letter I. The identity, it should be noted, is its own inverse, so that $I = I^{-1}$ [6].

We can also compose mappings, so that if a transformation S carries a point p to p' and a transformation T carries a point p' to point p'', so that the mapping ST carries a point p to point p''. It should be noted that the composition of the mappings is generally not commutative, so the mappings ST and TS will not always produce the same result, depending on what types of transformations T and S are [6].

We can also compose mappings with their inverses (the inverses are usually written such that the inverse of transformation S will be transformation S^{-1}, as we have written above). When we compose a mapping with its inverse (which is also a transformation), the result is the identity, so that $SS^{-1} = I$. The composition of a mapping with its inverse is usually commutative, so that $SS^{-1} = S^{-1}S = I$. As noted above, the identity is its own inverse; the transformation of reflection over a line is also its own inverse, since the double iteration of reflection SS will return the point p to itself, and so $SS = I$ [6].

A symmetry (also called an automorphism or similarity) is defined by Weyl [6] as "those transformations that leave the structure of space unchanged". Weyl [6] goes on to state that, "Given a spatial configuration $\mathfrak{F}$, those automorphisms of space which leave $\mathfrak{F}$ unchanged form a group Γ and *this group exactly describes the symmetry possessed by* $\mathfrak{F}$". This means that, given a "spatial configuration", or in more common terms a shape (represented as $\mathfrak{F}$), the group of $\mathfrak{F}$ are those automorphisms which leave the shape unchanged, that is, "appearing the same". Since the group includes all transformations which leave the shape "appearing the same", the identity I would also belong to this group, since it maps every point onto itself.

The example that Weyl [6] gives is that of a pentagram (Figure 3) where he notes that the pentagram possesses 5 distinct lines of reflection, and possesses 5-fold rotational symmetry. These symmetries form the group to which the pattern belongs. However, other patterns can also belong to this symmetry group, such as a pentagon or other shapes possessing the same symmetries. This shows that different patterns, even those that appear quite different, can belong to the same group. From this, we can begin theorizing about the number of groups needed to classify all "configurations of space" within certain constraints (the number of dimensions, for example).
We then consider the case of frieze patterns, which we have defined as repeating patterns that exist in the two-dimensional plane, but repeat infinitely in only one dimension. Within these constraints, experimentation can reveal that all frieze patterns belong to one of seven groups, characterized by translations, reflections, rotations, and glide reflections. We will eventually prove why this is the case, but we must first examine the case of ornamental patterns, which we have defined as one-dimensional patterns that repeat in only one direction; we theorize that there are only two of these.

2 Mathematical Perspective

2.1 Symmetry Groups of Ornamental Patterns

We begin with five examples of ornamental patterns, which we model as strings of beads (see Figure 4 below). Infinitely many such patterns can be constructed by introducing beads of varying sizes along the string.

As the simplest example, consider a pattern composed of beads all of the same size. This can be interpreted as a single bead repeated infinitely through translation, forming a necklace. Such a pattern exhibits translational symmetry. In addition, it possesses two instances of reflection symmetry: one at the center of each bead and another midway between any two consecutive beads.

In the second example, the pattern consists of a repeating unit made up of one small bead followed by one large bead. This unit is repeated infinitely in both directions to form a necklace. Like the previous example, this pattern has translational symmetry. Also like the previous example, it has two distinct sets of reflection points: one at the center of each small bead and another at the center of each large bead.

The third example that we consider is a necklace made of repetitions of a small bead, a medium bead, and a large bead. This necklace has a translational symmetry but no lines of reflection.

The fourth example we examine is that of a large bead followed by a medium and small one. It is similar to the third example in that it lacks reflective symmetry.

We can also extend the units that are repeated much further, as in example five, where we examine the patterns of beads on a Catholic rosary, where the repeating unit is a "decade" of beads, consisting of 10 small beads followed by one large bead. This pattern, too, includes two distinct instances of reflectional symmetry in the center of each large bead and in between the 5th and 6th small beads.

Patterns 1, 2, and 5 each have the same symmetries — they all consist of translation with two distinct instances of reflection.

By examining these and similar patterns, we can propose a classification of all ornamental patterns into a finite number of distinct ornamental groups. We formulate the statement in the form of a proposition.

Proposition 1 *There exist two distinct ornamental groups, one consisting of translation, and the other consisting of both translation and reflection.*

Proof: There are only two transformations that are possible in strictly one dimension: translation and reflection. All ornamental groups must therefore consist of some combination of these transformations.

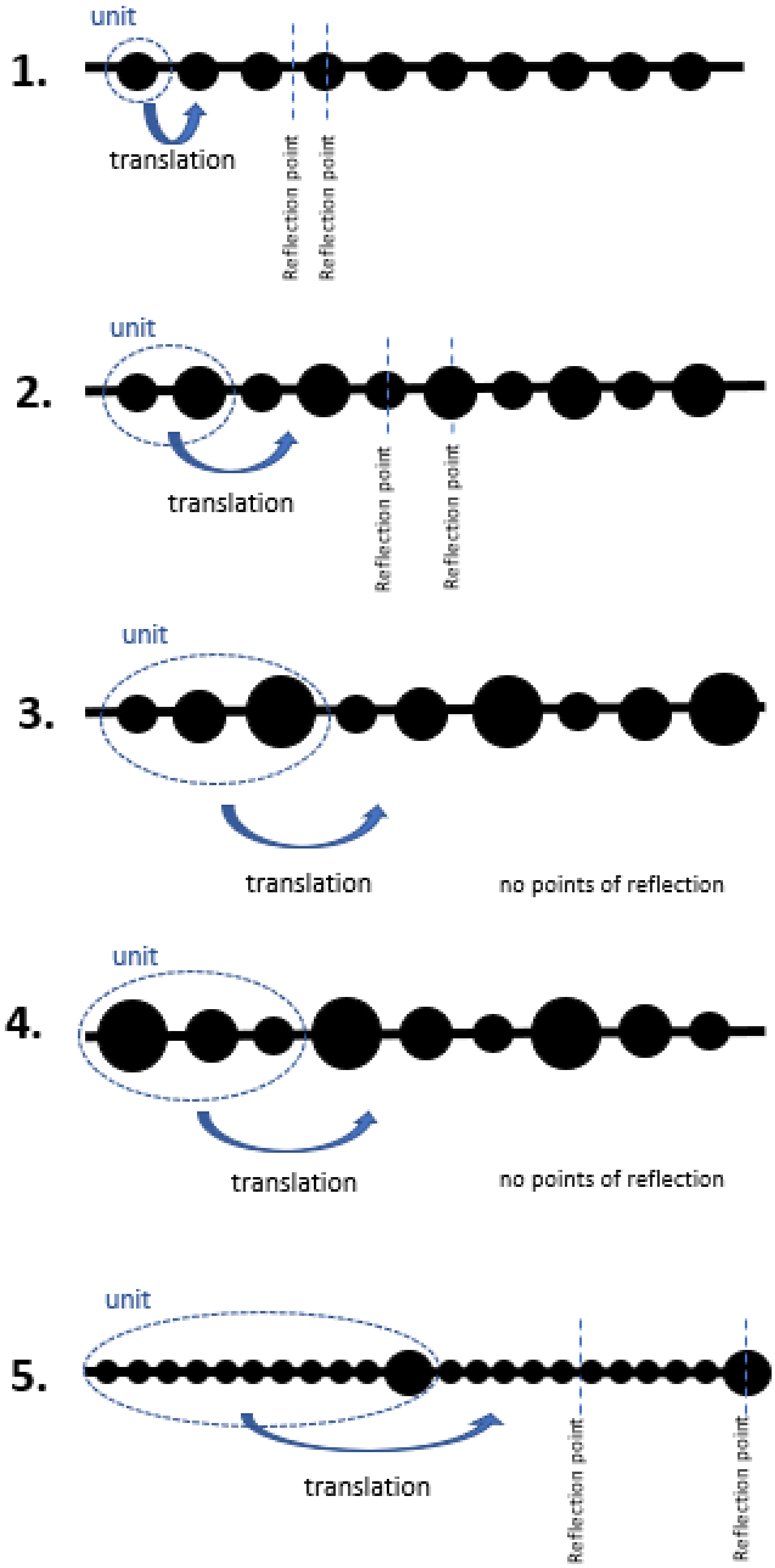

Figure 4: Some examples of ornamental patterns with identified units and symmetries.

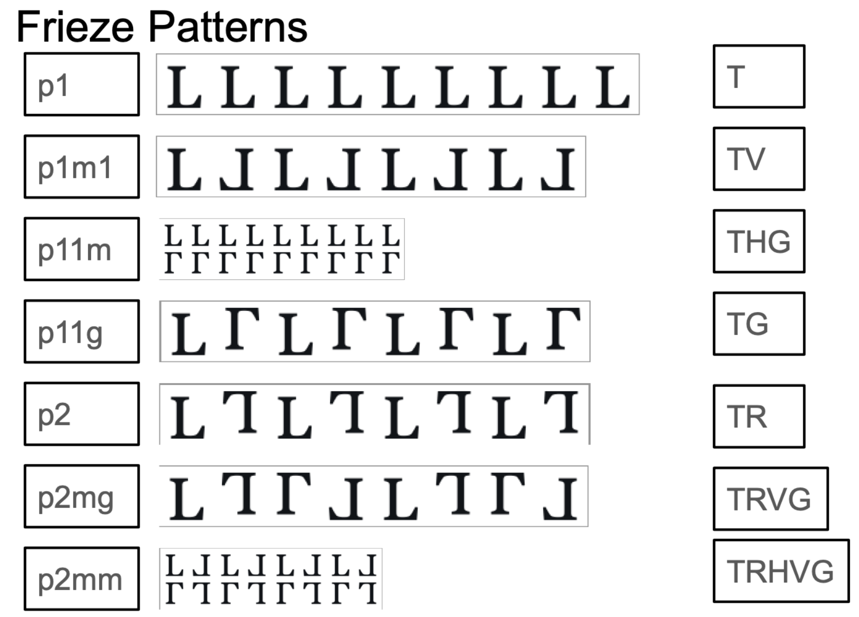

Figure 5: Illustration of all distinct frieze patterns.

Now, we analyze the possible groups that can be formed using these symmetries: (i) a group with only translation is valid and represents one type of ornamental group; (ii) a group with translation and reflection is another valid combination, and it forms a different type of ornamental group.

However, a group with only reflection is not valid for ornamental patterns, because a reflection alone does not generate a repeating pattern. Any ornamental pattern must involve translation, since repetition is inherent to ornamentation. □

2.2 Symmetry Groups of Frieze Patterns

There are seven types of frieze patterns, as illustrated in Figure 5. Before proceeding to state the theorem and provide its proof, we describe each pattern. **Pattern p1** involves translation only and is also considered an ornamental pattern. **Pattern p2** features two distinct points of two-fold (or 180°) rotational symmetry. **Pattern p11g** includes a longitudinal line of glide reflections. **Pattern p1m1**, like p1, is based on translation but also includes two distinct vertical lines of reflection, making it another ornamental pattern. **Pattern p11m** includes reflection across a longitudinal (horizontal) line. **Pattern p2mg** combines longitudinal glide reflection with two vertical lines of reflection, resulting in distinct points of rotation as well. **Pattern p2mm** is similar to p2mg, but instead of a glide reflection, it has a longitudinal line of ordinary reflection.

According to Schattschneider [7], the names of the frieze patterns are read from left to right and follow a convention established by the International Union of Crystallography. The initial letter "**p**" stands for a **primitive cell**, which refers to the smallest unit that repeats throughout the pattern. Following the "**p**", a number indicates the highest order of rotational symmetry present. Four of the seven frieze patterns lack rotational symmetry and are therefore labeled with a "**1**". The remaining three patterns do exhibit rotational symmetry and are labeled with a "**2**", indicating second-order rotational symmetry—that is, a 180° rotation. The third character signifies the presence or absence of vertical reflection axes (i.e., a series of vertical lines of reflection perpendicular to the horizontal axis). An "**m**" denotes mirror (reflection) symmetry, a "**g**" indicates glide reflection symmetry, and a "**1**" signifies the absence of vertical reflection symmetry. The fourth symbol, when included, describes the presence of a horizontal reflection axis, using the same notation ("**m**" for mirror, "**g**" for glide, "**1**" for none). If a pattern has only three characters, it lacks a horizontal reflection axis.

Further, to prepare for the theorem on the number of frieze patterns, we begin by stating and proving the following proposition.

Proposition 2 *There is always a translational symmetry in a frieze pattern.*

Proof: We examine each of the five symmetries in the context of two-dimensional frieze patterns. Translation clearly involves translational symmetry. Glide reflection also includes translation, as it consists of a reflection followed by a translation of the reflected unit. Reflection over vertical lines implies translation as well, since each reflection is undone by the next, effectively shifting the pattern; thus, the reflected unit is half the length of the translated unit. The same logic applies to rotational symmetry—a rotation followed by another results in a translation.

This leaves reflection over a horizontal line as the final transformation to consider. A pattern whose only symmetry is reflection over a horizontal line cannot be classified as a frieze pattern, because it lacks the defining feature of infinite translational repetition. To possess this property, such a pattern would need to include at least one of the other four symmetries, each of which inherently involves translation. $\square$

Now we are ready to prove the theorem.

Theorem. There exist a total of seven frieze groups.

We take the proof from the paper of Belcastro and Hull [8].

Proof: Having established that every frieze pattern possesses translational symmetry, we now consider the remaining four transformations that can contribute to additional symmetries. These are: rotation by 180 degrees, horizontal reflection (across the longitudinal axis), vertical reflection, and glide reflection. From these four, we can select any subset—including none—to combine with translation, yielding:

$$\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 1 + 4 + 6 + 4 + 1 = 16$$

possible combinations of symmetries [8]. These include the groups: t, r, h, v, g, hv, hr, hg, vr, vg, rg, hvr, hvg, hrg, vrg and $hvrg$, where t denotes translation, r denotes 180° rotation, h denotes horizontal reflection, v denotes vertical reflection, and g denotes glide reflection. The t for translation is omitted in all groups but the first one because we have already proven that translation is in every frieze group. The exception is for the first group, because translation is the only symmetry of that group.

However, many of these combinations are not distinct frieze groups because certain combinations imply the presence of others. For example, if a pattern has both horizontal and vertical reflectional symmetry, it necessarily also has 180° rotational symmetry, and vice versa. Therefore, the group with both horizontal and vertical reflections is equivalent to the group that includes all three symmetries: hvr.

Another example is the combination of translation and horizontal reflection. Such a pattern must also exhibit glide reflection, since applying a horizontal reflection followed by a translation parallel to the reflection line defines a glide reflection. Thus, this combination corresponds to the group hg.

We can continue this process of identifying equivalent sets of transformations and grouping them accordingly, thereby reducing the original sixteen combinations to just seven distinct frieze groups. A detailed explanation of this classification can be found in the article "Classifying Frieze Patterns Without Using Groups" by Belcastro and Hull [8], from which this proof is adapted.

Table 1 listing all sixteen symmetry combinations, along with explanations for why certain combinations are not counted as distinct, is reproduced from their paper below.

Table 1: List of All Possible Frieze Groups.

t		~~hr~~, ~~hvr~~, ~~hrg~~	must also have v
~~h~~	must also have g	~~vr~~	must also have g
v		~~vg~~	must also have r
r		~~rg~~	must also have v
g		vrg	
~~hv~~, ~~hvg~~	must also have r	$hvrg$	
hg			

□

3 Directions for Future Research

Further research in the area of symmetry groups can be done in two directions: diving deeper into understanding frieze patterns, and progressing to understanding more complicated symmetry patterns in higher dimensions. Both of these paths are made easier by an understanding of group theory, especially as it relates to symmetry groups. Group theory allows the creation of more rigorous proofs of the number of possible symmetry groups for a given space.

4 Supplemental Materials

The Python code used to generate the frieze pattern illustrations in this paper is available on GitHub at (https://github.com/Basileus1444/frieze-patterns). The author of the text fully encourages users to adapt the program for their own use.

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