Stochastic Models for Stock Option Pricing: Theory and Applications

Virat Varada
University High School,
Irvine, CA

Abstract

This paper analyzes option pricing using stochastic models, with a focus on European, American, and Asian options. Stock prices are modeled using Geometric Brownian Motion (GBM), which forms the basis for the Black–Scholes model and the Binomial Option Pricing Model. Python is used to identify stocks consistent with the GBM assumptions, and these stocks serve as case studies to illustrate the practical implementation of the models.

Keywords: European stock option, American stock option, Asian stock option, call option, put option, geometric Brownian motion, Black-Scholes model, binomial option pricing model

1 Introduction

1.1 Background

This paper examines stochastic models to determine the fair value of stock options. A stock option is a financial contract that grants its holder the right (but not the obligation) to buy (call option) or sell (put option) an underlying stock at a predetermined price, known as the strike price, before or on a specified expiration date. Because option values depend on factors such as volatility of the stock price, time to expiration, and interest rates, their pricing can be complex. To address this, models such as the Black–Scholes model, based on continuous-time mathematics, and the Binomial Option Pricing model, using discrete-time steps, have been developed. These models not only facilitate accurate option pricing but also help investors manage risk and understand the dynamics of options better. In this study, the historical datasets from the Yahoo Finance website are analyzed using Python to identify stock price behavior and to apply these pricing models in practical case studies.

1.2 Literature Review

Stock options are contracts that give the right to buy or sell a stock at a set price by a certain date. There are different types of options with varying exercise rules and payoff structures. For both American and European options, the payoff depends on the strike price. European-style options can only be exercised at the expiration date, while American options can be exercised at any time prior to expiration. Exotic options, such as Asian options, differ from vanilla American and European options because their payoff is determined by the average price over a certain time period rather than the price at expiration.

The Black-Scholes model was created in 1973 by Fischer Black, Robert Merton, and Myron Scholes [1]. It is a well-known model used to price European options under the assumption of geometric Brownian motion for stock prices.

The Binomial Option Pricing Model can be applied to both American and European options. Benninga and Wiener (1997) show that the binomial option pricing model converges to the Black-Scholes formula in the limit as the number of steps increases, for European options and certain American options where early exercise is not optimal, such as calls on non-dividend-paying stocks [2]. They also discuss the pricing of exotic options, including path-dependent options like Asian options, using binomial trees with backward induction, demonstrating that American-style exotic options are generally more valuable than their European counterparts due to the early exercise feature.

Fadugba et al. (2014) assess the performance of the binomial model for pricing American and European options, emphasizing its mathematical simplicity, reliance on no-arbitrage assumptions, and flexibility in handling early exercise [3]. They derive the Black-Scholes equation using the binomial approach under risk-neutral valuation and highlight that the model approximates continuous-time prices as the number of steps increases, making it particularly suitable for American options without closed-form solutions.

Dar and Anuradha (2018) compare the binomial model and the Black-Scholes model for pricing European call and put options, using statistical tests such as the t-test and Tukey test at one period [4]. Their findings indicate no significant difference in the mean prices produced by the two models, suggesting comparable accuracy for European options, though the binomial model is noted for its simplicity.

2 Geometric Brownian Motion: Theory

A standard Brownian motion $\{B(t), t \geq 0\}$ is a stochastic process that satisfies $B(0) = 0$, has stationary and independent increments, and for $t > 0$, $B(t)$ follows a normal distribution with mean 0 and variance t. Louis Bachelier first introduced the mathematics of Brownian motion and applied it to model stock and option prices [5]. However, the standard Brownian motion model has several limitations when applied to financial markets, such as the possibility of negative prices and the assumption of independent price movements.

A stochastic process $\{Y(t), t \geq 0\}$ is called a geometric Brownian motion (GBM) if it satisfies the following stochastic differential equation:

$$dY(t) \;=\; \mu Y(t)\, dt \,+\, \sigma\, Y(t)\, dB(t),$$

where μ is the drift coefficient, σ is the volatility coefficient and $B(t)$ is a standard Brownian motion. [insert reference or two] The solution to this equation is as follows:

$$Y(t) \;=\; Y(0) \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) t \,+\, \sigma\, B(t)\right].$$

Geometric Brownian motion is widely used to model stock prices, although it relies on simplifying assumptions, such as constant drift and volatility, which can sometimes reduce its accuracy in real market applications.

3 Geometric Brownian Motion: Applications

To illustrate, historical stock price data between April 2023 and January 2025 were downloaded from Yahoo Finance (https://finance.yahoo.com), and a Python program was used to simulate stock prices using the GBM model. Apple Inc. (ticker: AAPL), a leading American technology company known for its consumer electronics and software, and SAP SE (ticker: SAP.DE), a major European enterprise software company headquartered in Germany, were selected as examples of American and European stocks, respectively. Each data set contains 432 rows and two columns, date and stock price at the closing of the stock market.

To estimate the drift coefficient μ and volatility σ, we write the increments of the natural logarithm of the prices as

$$\ln Y(t_i) - \ln Y(t_{i-1}) = \mu (t_i - t_{i-1}) + \sigma \big(B(t_i) - B(t_{i-1})\big).$$

Taking unit length time increments, $t_i - t_{i-1} = 1$, and using the stationarity of increments, we see that the log ratios (or log price increments)

$$\ln Y(t_i) - \ln Y(t_{i-1}) = \ln \frac{Y(t_i)}{Y(t_{i-1})}$$

are distributed as $\mu + \sigma B(1)$, which follows a normal distribution with mean μ and variance σ^2. Therefore, the mean and variance of the sample serve as natural estimators for μ and σ^2, respectively. The estimates for both datasets are summarized below.

Data	$\hat{\mu}$	$\hat{\sigma}$
Apple Inc.	0.2062	0.2140
SAP SE	0.4695	0.2138

Finally, we plot both the observed trajectories and 100 simulated trajectories generated from the estimated parameters. The plots for each stock appear in Figures 1 and 2.

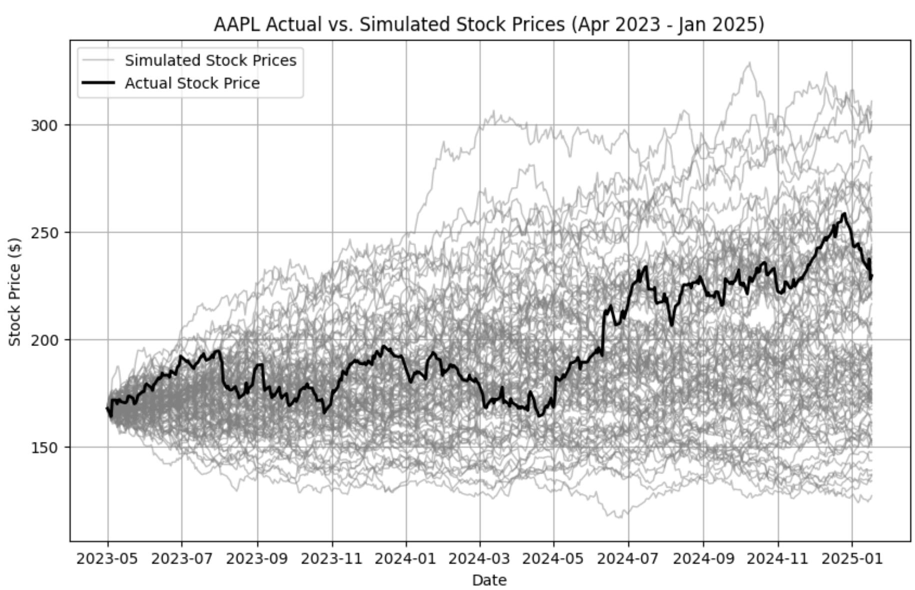

Figure 1. Actual and Simulated GBM Trajectories of Apple Stock Closing Prices.

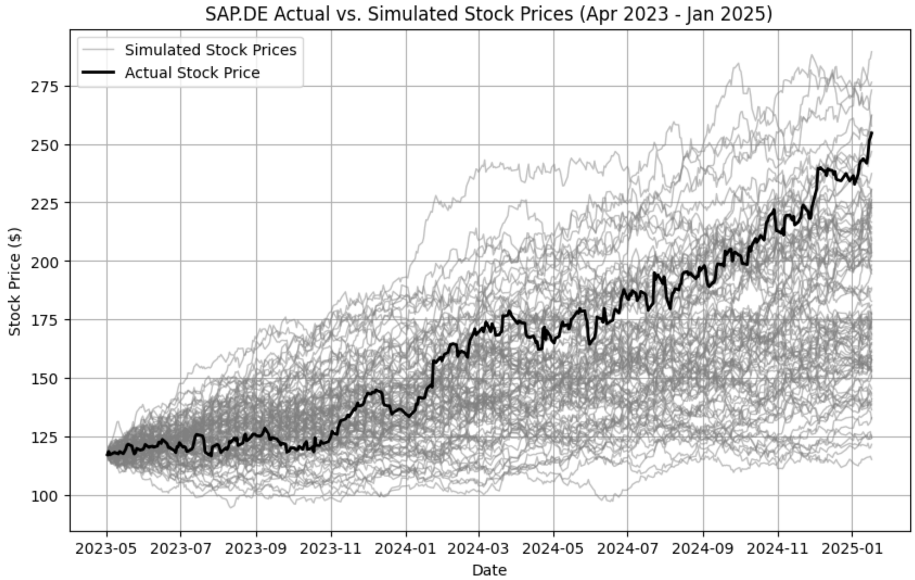

Figure 2. Actual and Simulated GBM Trajectories of SAP SE Stock Closing Prices.

4 European Stock Option Pricing: The Black–Scholes–Merton Model

The Black–Scholes–Merton Option Pricing Model is one of the most influential results in financial mathematics, providing a theoretical price for a European option under the assumption that the price of the underlying security follows geometric Brownian motion. A European option is a financial contract that can only be exercised on its expiration date. The model applies to both call and put options: a call option gives the holder the right to buy the stock at a specified price, while a put option gives the holder the right to sell it at that price.

The model was first derived in 1973 by Fischer Black and Myron Scholes and was further developed by Robert Merton in the same year [6]. For this work, Scholes and Merton were awarded the 1997 Nobel Prize in Economics. Black, who had passed away in 1995, was not eligible to share the award, since Nobel Prizes are not given posthumously.

To introduce the model, let us define the key quantities:

$$\begin{aligned} S &: \text{ the current stock price,} \\ K &: \text{ the strike (exercise) price,} \\ T &: \text{ the time to expiration,} \\ r &: \text{ the constant risk-free interest rate,} \\ \sigma &: \text{ the volatility of the stock price,} \\ N(x) &: \text{ the cumulative distribution function of the standard normal distribution.} \end{aligned}$$

With these, we define

$$d_1 = \frac{\ln(S/K) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}.$$

The Black–Scholes model relies on several assumptions: S follows geometric Brownian motion, r is constant, the stock pays no dividends, the markets are frictionless, there are no arbitrage opportunities, and investors can short-sell and trade fractional shares. Under these conditions, the option price V satisfies the Black–Scholes partial differential equation (PDE):

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV.$$

Solving this PDE yields the celebrated Black–Scholes formulas for European options:

$$\begin{aligned} \text{Call option:} \quad & C = SN(d_1) - Ke^{-rT}N(d_2), \\ \text{Put option:} \quad & P = Ke^{-rT}N(-d_2) - SN(-d_1). \end{aligned}$$

This result, which unified probability theory, stochastic processes, and financial economics, revolutionized modern finance and remains the foundation of option pricing theory today.

5 American Stock Option Pricing: The Binomial Model

Unlike European options, which can only be exercised at expiration, American options allow the holder to exercise the option at any time prior to the expiration date. Since the Black–Scholes model assumes exercise only at expiration, it does not directly apply to American options. To address this, the Binomial Option Pricing Model was developed by John Cox, Stephen Ross, and Mark Rubinstein in 1979 [7]. Like the Black–Scholes model, it assumes that the stock price follows geometric Brownian motion, but it introduces a discrete-time framework that naturally accommodates early exercise.

In the binomial model, option prices are computed iteratively using a binomial tree. From a current stock price S, the stock can either move up by a factor u, or down by a factor d in each time step of length $\Delta t = T/N$, where T is the expiration time and N is the number of steps. These factors are defined as

$$u = e^{\sigma\sqrt{\Delta t}}, \text{ and } d = \frac{1}{u} = e^{-\sigma\sqrt{\Delta t}},$$

where σ is the volatility of the stock. After one step, the possible stock prices are $S_u = S \cdot u$ and $S_d = S \cdot d$. Extending this to n steps, if the stock moves up j times and down $(n - j)$ times, the price becomes

$$S_{n,j} = S\, u^j d^{n-j}.$$

At maturity $(t = T)$, the option payoff is determined by the strike price K:

$$\text{Call: } \max(S_{n,j} - K, 0), \qquad \text{Put: } \max(K - S_{n,j}, 0).$$

For earlier times $t = (N - 1)\Delta t, \ldots, 0$, the option value at each node is given by the maximum of: 1. The exercise value, corresponding to exercising the option immediately, and 2. The hold value, which is the discounted expected value under the risk-neutral measure.

The risk-neutral probability of an up move is

$$p = \frac{e^{r\Delta t} - d}{u - d},$$

where r is the risk-free interest rate. Thus, the option value at time t and node i (corresponding to i up moves) is

$$V_{t,i} = \max\big(\max(Su^i d^{t-i} - K, 0), e^{-r\Delta t}\left[pV_{t+\Delta t,\, i+1} + (1-p)V_{t+\Delta t,\, i}\right]\big) \quad \text{(call)}.$$

For a put option, the immediate exercise value is $\max(K - S\, u^i d^{t-i}, 0)$. Iterating backward from $t = T$ to $t = 0$ yields the option price $V_{0,0}$.

The binomial model is highly flexible and well-suited for valuing American options because it explicitly incorporates the possibility of early exercise. Moreover, as $N \to \infty$, the binomial model converges to the Black–Scholes price, establishing a deep connection between the two approaches. Although the basic model assumes that the stock pays no dividends, extensions can incorporate dividend payments as well.

6 European and American Stock Price Options: Applications and Results

In this section, we apply option pricing models to Apple (AAPL) and SAP SE (SAP.DE). Both models, the Black-Scholes framework and the binomial option pricing model, assume that stock prices evolve according to a geometric Brownian motion. Since our earlier analysis confirmed this assumption for Apple and SAP through simulated trajectories, we can justifiably apply the Black–Scholes model to SAP and the binomial model to Apple.

For SAP, we use option chain data and implement the Black-Scholes formula, which incorporates the observed strike price, current stock price, time to maturity, risk-free rate, and estimated volatility. The procedures for estimating volatility and the risk-free rate follow the same approach used in our geometric Brownian motion analysis. The model-generated prices are then compared with market quotes to assess the precision of the Black-Scholes approximation. Based on one year of SAP option chain data, we find that about 25.85% of contracts are exercised, while the remainder expire out of the money.

The binomial model is flexible enough to handle both European and American options, so we apply it to Apple's American-style options. Using one year of AAPL option chain data, we estimate that approximately 25.50% of contracts are exercised. See Figure 3 below. These results suggest that while the binomial model is essential for valuing American options, the Black–Scholes framework can also deliver reliable pricing for American-style contracts in many cases.

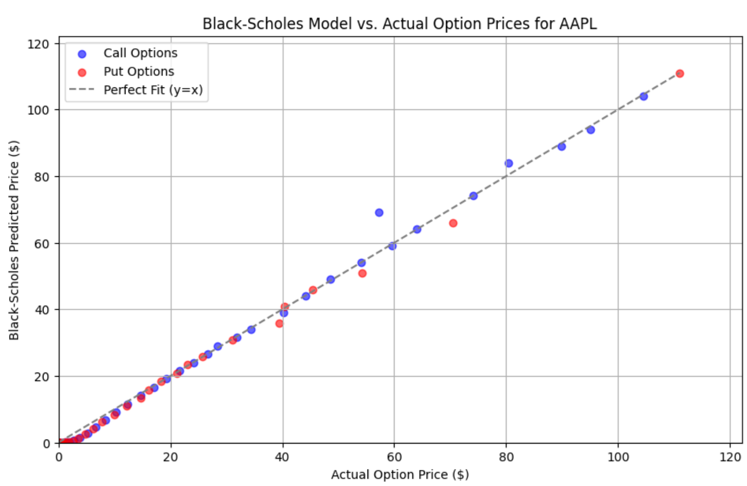

Figure 3. Black-Scholes vs. actual option prices for Apple Stock.

7 Conclusion and Discussion

7.1 Summary

This paper examines the pricing and exercise behavior of stock options under a stochastic framework grounded in geometric Brownian motion (GBM). We reviewed the Black–Scholes–Merton model for European options and the binomial option pricing model for American options, emphasizing the convergence of the binomial lattice to Black–Scholes prices for European claims and the lattice's flexibility in handling early exercise.

On the empirical side, we implemented Python programs that queried Yahoo Finance for option chain data and market inputs, addressed time-stamp consistency, and then computed risk-neutral exercise probabilities. For SAP (treated as European-style), we used the Black–Scholes $N(d_2)$ for calls and $N(-d_2)$ for puts to obtain the probability that the option finishes in-the-money at expiration. For AAPL (American-style), we built a CRR binomial tree with dividend yield to determine the optimal stopping rule and the probability of being exercised at or before expiration. We summarized both unweighted and open-interest-weighted (OI-weighted) exercise probabilities across maturities and strikes. Consistent with market intuition, the probability of exercise increases with moneyness, often hovers near 50% for near-the-money short-dated options, and is typically higher for deep-in-the-money contracts. The aggregated results over the one-year sample indicate that roughly one quarter of listed contracts are expected to be exercised under the risk-neutral measure (about 25.85% for SAP and 25.50% for AAPL), which aligns with the empirical reality that most listed options expire out-of-the-money.

Several practical considerations temper these findings. First, the risk-neutral exercise probabilities quantify model-implied, not realized, frequencies and do not capture behavioral or institutional features of exercise (e.g., early assignment conventions). Second, GBM with constant volatility and rates is a simplification; volatility smiles/skews and discrete dividends can affect both pricing and exercise decisions. Third, data availability (e.g., ADR vs. local listing, missing implied volatilities) and our use of proxy risk-free rates introduce measurement error. Nonetheless, the pipeline demonstrates a coherent approach for connecting theory to data and producing interpretable, model-based estimates of exercise likelihoods.

7.2 Future Directions

A natural extension is to apply Monte Carlo simulation methods to the valuation and risk analysis of exotic options, particularly those with path-dependent payoffs:

- Asian options (arithmetic and geometric average payoffs): simulate GBM paths and compute discounted payoffs based on running averages; use control variates (geometric Asian with a closed-form solution) and antithetic variates to reduce variance [8].
- American-style exotics: use Least-Squares Monte Carlo (LSMC) to learn continuation values and optimal stopping rules along simulated paths, enabling early exercise under complex payoff structures [9].
- Barrier and lookback options: incorporate path-dependent state tracking (e.g., running extrema or barrier crossings) within the simulation; employ Brownian-bridge corrections to better capture barrier hits between time steps [10].

Beyond Monte Carlo for exotics, several enhancements can increase realism and robustness: calibrating to implied volatility surfaces and testing alternative dynamics (local volatility models, stochastic volatility models such as Heston, or jump-diffusion processes), modeling discrete dividends and ex-dividend dates explicitly (crucial for early-exercise calls), and extending to multi-asset options via correlated path simulation (using Cholesky decomposition). From a computational standpoint, quasi-Monte Carlo (e.g., Sobol sequences), parallelization, or GPU acceleration can materially reduce estimator variance and runtime [11, 12]. Finally, integrating backtesting of hedging performance and conducting out-of-sample validation against traded prices would provide a more complete assessment of model accuracy and practical applicability.

Supplemental Materials

All datasets and Python code used in this study are available on Github

https://github.com/mathvv1234/-Stock-Option-Pricing

for reproducibility and further exploration.

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