Exploring Mathematical Structures in Music

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Abstract

Bela Bartok encoded the Fibonacci sequence into the structure of his sonatas. Along with him, countless other legendary composers found ways to weave the fine concept of mathematics into each measure of their work. This essay captures just a few masterful and mathematical concepts that are often overlooked but powerful in classical music and some timeless composers who have used mathematics to define and lay a foundation for their music.

Keywords: music, mathematics, Fibonacci sequence, aesthetic value of music, harmony, reflection, translation

1 Introduction

In music, there exists a rich interplay between sound and mathematics, with countless mathematical concepts embedded within even the most fundamental aspects of musical theory. At the most basic level, arithmetic is present in elements such as time signatures, note durations, and rhythmic patterns, where fractions dictate how beats are divided and combined to form rhythm. Scales and intervals also reveal numerical relationships, as the spacing between notes often follows proportional or geometric patterns. Moving beyond these foundational concepts, more sophisticated ideas emerge in the form of tuning systems, such as equal temperament or just intonation, which rely on ratios and logarithmic calculations to determine pitch. Even chord progressions and harmonics can be analyzed mathematically, uncovering patterns and symmetries that give music its sense of balance and resolution. While many listeners may not consciously recognize these underlying structures, they are ever-present, forming the hidden framework that shapes the sound and emotion of music. From simple fractions to complex logarithmic relationships, mathematical principles in music span a spectrum of complexity, illustrating the deep and often overlooked connection between numbers and art. Mathematics can be found in ratios in harmonies, the Fibonacci sequence to determine the melody, and geometric translations within music building. Nevertheless, there are also wrong ways to connect these two similar concepts, as shown on numerous occasions.

However, there is a fine line between true and false connections between mathematics and music. A well-renowned mathematician from the early twentieth century, George David Birkhoff (1884-1944), sought to establish a general mathematical theory of aesthetics that could be applied across various forms of art, including music. In his 1933 work *Aesthetic Measure*, Birkhoff proposed that the aesthetic value of an artwork could be quantified using the formula O/C, where O represents the order, or the degree of structure, symmetry, and regularity in the piece, and C measures the complexity, which corresponds to the effort required to perceive or understand the work. [1] According to Birkhoff, a piece with high order but low complexity would maximize aesthetic appeal, while high complexity would diminish it, as the mental effort needed to decode the work detracts from its immediate perceptual enjoyment.

Despite its pioneering nature, Birkhoff's theory has been widely debated on both qualitative and quantitative grounds. [2] Critics have questioned the validity of reducing aesthetic experience to a single formula, noting that artistic value may depend on subjective, cultural, and contextual factors that cannot be captured numerically. Even the formula itself has been critiqued: some argue that alternative mathematical relationships, such as $O - C$ or O/C^2, might better model the interaction between order and complexity. Furthermore, the relative weights assigned to order and complexity are not universal, raising questions about whether the same formula can meaningfully apply to different art forms or individual experiences. Nonetheless, Birkhoff's attempt represents an early and influential effort to bridge mathematics and aesthetics, highlighting the intriguing connections between numerical structure and human perception of beauty.

Figure 1: George David Birkhoff

2 Harmonies and the Math Behind It

Harmony, a central element of music, arises when two or more pitches are sounded simultaneously, creating a combination of tones that can be pleasing, tense, or expressive. At its most basic, harmony takes the form of a chord—a set of notes played together that typically follows a defined pattern of intervals. The mathematical foundations of harmony lie in the ratios between the frequencies of these notes. For instance, the octave corresponds to a frequency ratio of 2:1, a perfect fifth to 3:2, and a major third to 5:4. These simple integer ratios have been observed to produce consonance, the sense of tonal stability, whereas more complex ratios often result in dissonance. Beyond individual intervals, harmonic progressions—the sequence of chords in a piece—also reflect mathematical structures, as composers balance predictability and variation to create musical tension and resolution. By examining the numerical relationships in harmony, we gain insight into why certain combinations of sounds are universally perceived as pleasant, revealing a deep connection between mathematics and musical aesthetics. [3,4,5]

3 The Fibonacci Sequence in Musical Form

All composers have their own unique techniques and stylistic approaches to composition, yet many have been drawn to mathematical patterns such as the Fibonacci sequence and its related concept, the Golden Ratio. The Fibonacci sequence is defined recursively, where each term is the sum of the two preceding ones (1, 1, 2, 3, 5, 8, 13, ...). This mathematical progression has found applications not only in nature and art but also in the temporal and structural design of music. Composers such as Béla Bartók (1881-1945) and Claude Debussy (1862-1918) are often cited as having consciously or intuitively incorporated Fibonacci proportions into their compositions. [6,7]

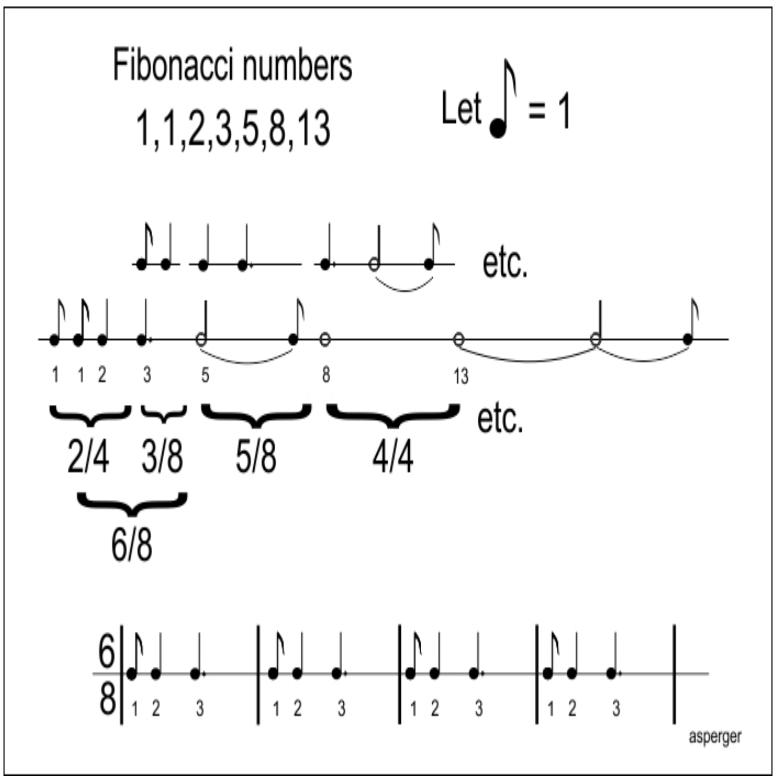

Figure 2: Visualization of the Fibonacci numbers as applied to musical composition.

In music, the Fibonacci sequence can be used to determine the relative lengths of notes, rhythmic groupings, or structural proportions within a piece. For example, when constructing rhythmic phrases, a composer might let the duration of a new note equal the sum of the durations of the two preceding notes—mirroring the recursive nature of the

Fibonacci pattern. If two eighth notes precede a quarter note, their combined rhythmic value (1/8 + 1/8 = 1/4) leads naturally into the next note being a quarter. The following note might then be a dotted quarter (1/4 + 1/8 = 3/8), and so on, generating an evolving rhythmic motif guided by the Fibonacci rule. This recursive pattern creates an organic sense of expansion and balance, aligning with the aesthetic principles that many associate with natural growth and proportion.

Bartók, in particular, often structured his musical forms and phrase lengths around Fibonacci ratios. Analytical studies of his works, such as Music for *Strings, Percussion and Celesta* (1936), reveal that climactic points frequently occur at or near the Golden Ratio point of a movement—approximately 0.618 of the total duration. [8] Similarly, Debussy's use of subtle proportional relationships in pieces like *La Mer and Reflets dans l'eau* ("Reflections in the water", 1905) demonstrates how Fibonacci-based timing and phrasing can yield a sense of natural fluidity and aesthetic balance. [7]

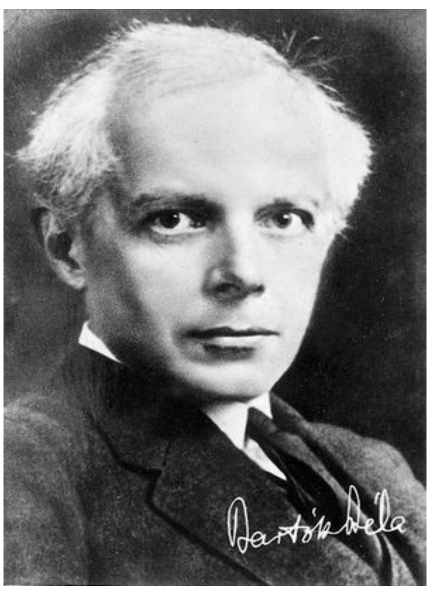

Figure 3: Béla Bartók

Figure 4: Claude Debussy

4 Geometric Reflections and Translations in Musical Structure

The compositional architecture of music often mirrors the logical and structural rigor found in mathematics. In particular, the concepts of geometric **reflection** and **translation** provide a useful framework for understanding how melodies and motifs are developed, transformed, and repeated within a piece.

A geometric reflection in music occurs when a melody is inverted around a central tonal axis. Each ascending interval in the original phrase is mirrored by a corresponding descending interval of the same magnitude, producing an "inverted" version of the melody. This is analogous to reflecting a geometric object across a line of symmetry, preserving spatial relationships while reversing direction. Johann Sebastian Bach (1685-1750) frequently employed this technique, particularly in his *The Art of Fugue* and *The Musical Offering*, where thematic material undergoes systematic inversion and retrograde. Such transformations not only demonstrate aesthetic symmetry but also exemplify the mathematical principles of reflection and reversal in the musical domain.

Figure 5: Johann Sebastian Bach

Similarly, geometric translation in music corresponds to the repetition of a melodic or rhythmic pattern at a different pitch or temporal position. When a motif is transposed to another key or shifted in time, it retains its structural integrity while occupying a new musical space. Camille Saint-Saëns (1835-1921) illustrates this concept effectively in his *Cello Concerto No. 1 in A minor, Op. 33.* The recurring motif, characterized by a dotted quarter note followed by a sequence of descending eighth notes, first appears in the opening measures of the concerto and reemerges in later movements, transposed and rhythmically altered. This translation of the motif across tonal and temporal dimensions creates unity through repetition, echoing mathematical notions of congruence and transformation.

Figure 6: Camille Saint-Saëns

Summary and Additional Considerations

The concept of mathematics in music is a highly absorbing topic that is commmonly underestimated. In music, mathematical concepts such as geometric translations along with the Fibonacci Sequence can be observed, especially in pieces of classical music. Many classical composers such as Johannes Bach and Bela Bartok are known to flexibly weave arithmetic into their works. However, there are numerous other topics in math that can be correlated with music that are not mentioned above, such as the use of logarithms in semitones. This concept can be further explored in professor David Wright's *Mathematics and Music*. [11]

Acknowledgments

I would primarily like to express my appreciation to Dr. Olga Korosteleva, a professor of statistics at California State University, Long Beach, for her mentorship, guidance, and dedication to this project. This project would not have been possible without her counsel and support. Additionally, I would like to thank my mother, Dr. Wan Lee, a professor of Chemical Engineering at CSULB, along with the rest of my family who made this project come together.

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