Modeling Stock Prices with Geometric and Fractional Brownian Motion

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Abstract

This paper analyzes the suitability of geometric Brownian motion, fractional Brownian motion, and geometric fractional Brownian motion for modeling stock price dynamics. Model parameters, including drift, volatility, and the Hurst exponent, are estimated using a designated training period for each asset. The accuracy of the model is then evaluated on testing-period data using the Kolmogorov–Smirnov test applied to standardized increments, which are expected to follow a standard normal distribution under the fitted models.

Keywords: Stock price, Brownian motion, drift, volatility, geometric Brownian motion, fractional Brownian motion, geometric fractional Brownian motion, Hurst parameter

1 Introduction

1.1 Background

Since the early 20th century, modeling price dynamics has been a central problem in financial mathematics. Standard Brownian motion (BM) is characterized by erratic, random linear fluctuations. While not optimal for modeling financial assets on its own, Brownian motion provides the essential framework that makes the other models possible.

A foundational model in this field is geometric Brownian motion (GBM), introduced by Osborne (1959) and popularized by Black and Scholes (1973). Geometric Brownian motion functions effectively as a "random walk." For example, the log-returns are normally distributed, and the past movements have zero correlation with the stock's future price movements.

While geometric Brownian motion is simple and mathematically useful in modeling the prices of certain volatile stocks, the model falls short in most real-world applications where past movements often do correlate with future prices. Empirical data have shown that stocks and other financial assets tend to exhibit heavy tails. For example, stable days tend to be followed by stable days, while volatile days are often followed by volatile days.

To address this dependence across time scales, researchers proposed fractional models such as fractional Brownian motion (fBM), introduced by Mandelbrot and Van Ness (1968). Although fBM is effective in modeling the prices of stable currencies, like GBM, it has limitations; specifically, fBM allows for cases where stock prices fall below zero, which is financially impossible. Finally, researchers proposed the geometric fractional Brownian motion model (GfBM). Unlike standard fractional Brownian motion, GfBM models price changes geometrically (i.e., multiplicatively) rather than arithmetically, ensuring the positivity of values and making it better suited to modeling standard growth companies than relatively stable currencies.

1.2 Literature Review

Classical continuous-time models of asset prices are rooted in Brownian motion (BM) and its exponential counterpart, geometric Brownian motion (GBM), which form the backbone of modern financial theory. GBM underlies the Black–Scholes framework and remains widely used due to its analytical tractability and consistency with the efficient market hypothesis, as it assumes independent and stationary increments. However, extensive empirical evidence has challenged the adequacy of pure Brownian dynamics, suggesting that financial time series may exhibit deviations from the Markov and short-memory assumptions. Early empirical studies on long-term dependence in asset prices and returns, such as those of Lo (1991), as well as the fractal market perspective advanced by Peters (1994), highlight the presence of scaling behavior and persistence across time horizons, motivating the exploration of alternative stochastic processes.

Fractional Brownian motion (fBM) extends BM by introducing a Hurst parameter that allows for long-range dependence and self-similarity, making it a natural candidate for modeling financial phenomena with memory effects. The mathematical properties of fBM and its relevance to finance have been studied extensively (Nourdin, 2012; Hu and Øksendal, 2003), and its use has been proposed in various contexts, including option pricing and volatility modeling (Necula, 2002; Comte and Renault, 1998). More recent work has further strengthened the case for fractional-type dynamics, particularly in volatility modeling, where empirical evidence suggests rough or persistent behavior consistent with fractional processes (Gatheral et al., 2018). More recently, Garcin (2022) explores forecasting applications based on fractional Brownian motion, providing empirical evidence that fractional dynamics can be useful for modeling and prediction in financial contexts.

1.3 Brownian Motion

A standard Brownian motion (BM) is a stochastic process used to model random behavior over time. A standard Brownian motion $B(t)$ starts at $B(0) = 0$ and has independent, normally distributed increments. Specifically, for $t > s$,

$$B(t) - B(s) \sim \mathcal{N}(0,\, t - s).$$

As the time interval $t - s$ increases, the variance grows linearly, resulting in a wider spread of possible outcomes. While Brownian motion forms the mathematical foundation for many models in finance, it has limitations when directly applied to stock prices, as it allows negative values and does not capture important empirical features such as volatility clustering or long-range dependence. Although standard Brownian motion will not be used as a stand-alone model in this paper, it serves as the framework for the models discussed below.

1.4 Geometric Brownian Motion

Geometric Brownian motion (GBM) modifies Brownian motion to ensure strictly positive values, making it more suitable for modeling stock prices. The process is defined as

$$X(t) \;=\; X(0)\, \exp\Big[(\mu - \frac{\sigma^2}{2})t + \sigma B(t)\Big],$$

where $X(t)$ is the price of the stock at time t, $X(0)$ is the initial price of the stock, μ is the drift, and σ is the volatility parameter.

Under GBM, the continuously compounded log-return over the interval $[0, t]$ is normally distributed with mean

$$\left(\mu - \frac{\sigma^2}{2}\right) t$$

and variance $\sigma^2 t$. This implies that log-returns have independent increments and a constant volatility parameter. While these features make the model analytically convenient, it often fails to capture key characteristics observed in real financial data, such as volatility clustering and temporal dependence.

To generate a trajectory of a GBM, we first generate a trajectory of a standard BM $B(t_1), \ldots, B(t_n)$, then compute the values of a GBM on the grid

$$X(t_k) \;=\; X(0)\, \exp\Big[\Big(\mu - \frac{\sigma^2}{2}\Big)t_k \;+\; \sigma\, B(t_k)\Big]$$

for some pre-determined strictly positive initial value $X(0)$. A continuous trajectory is formed by linking the simulated points with line segments. As an illustration, we generate a sample trajectory with $\mu = 0.7$, $\sigma = 0.3$, and $X(0) = 100$ (see Figure 1).

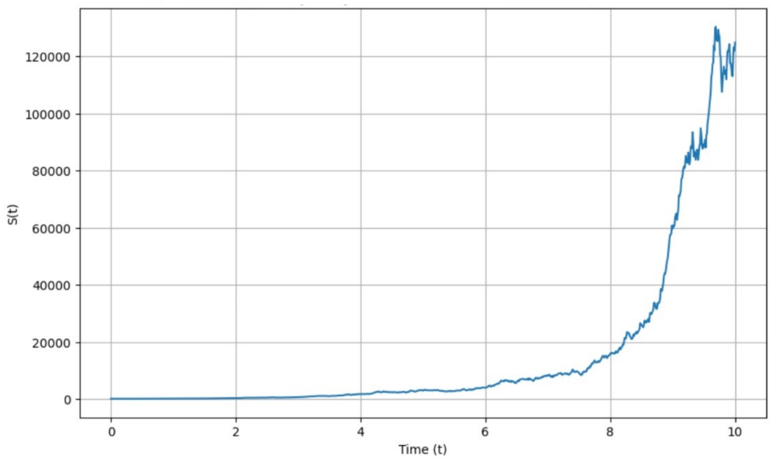

Figure 1: Sample trajectory of a geometric Brownian motion.

1.5 Fractional Brownian Motion

A fractional Brownian motion (fBM) is a generalization of the standard Brownian motion that incorporates dependence between increments. Unlike classical Brownian motion, which has independent increments, fBM exhibits *memory*, meaning past movements influence future changes.

A fractional Brownian motion is characterized by the Hurst parameter $H \in (0, 1)$, named for Harold Edwin Hurst, a hydrologist who studied water storage (Hurst et al, 1965).
- If $H = 0.5$, the process reduces to a standard Brownian motion.
- If $H < 0.5$, the increments are negatively correlated (rough, mean-reverting paths).
- If $H > 0.5$, the increments are positively correlated (persistent, smoother paths).

An fBM starts at zero, that is, $B^H(0) = 0$, and is a centered Gaussian process with zero mean, $\mathbb{E}(B^H(t)) = 0$ for all $t \geq 0$. For an fBM with volatility parameter σ, the variance and covariance are given by

$$\mathbb{V}\text{ar}\big(B^H(t)\big) = \sigma^2 t^{2H},$$

and

$$\mathbb{C}\text{ov}\big(B^H(t), B^H(s)\big) = \frac{\sigma^2}{2}\Big(t^{2H} + s^{2H} - |t-s|^{2H}\Big).$$

Unlike the standard Brownian motion, where the variance grows linearly in time, the variance of fBM grows nonlinearly as t^{2H}, reflecting its long-memory structure.

To generate a sample path of fractional Brownian motion $B^H(t)$ on a grid $t_1, \dots, t_n$ using independent and identically distributed standard normal random variables $Z_1, \dots, Z_n$, we begin by choosing a Hurst parameter $H \in (0, 1)$ and a volatility parameter σ, and then form the $n \times n$ variance-covariance matrix Σ with entries are given by

$$\Sigma_{ij} = \frac{\sigma^2}{2}\Big(t_i^{2H} + t_j^{2H} - |t_i - t_j|^{2H}\Big),$$

which corresponds to the covariance structure of the fractional Brownian motion. Next, we compute the Cholesky factorization $\Sigma = LL^\top$, where L is a lower triangular matrix, and define $B^H = LZ$, where $Z = (Z_1, \ldots, Z_n)^\top$. The vector B^H then has the desired covariance Σ, and its components provide a simulated realization of $B^H(t_1), \ldots, B^H(t_n)$, with $B^H(t_0) = B^H(0) = 0$ set by convention. A generated trajectory is given in Figure 2. We used $\sigma = 0.3$ and $H = 0.4$ for this simulation.

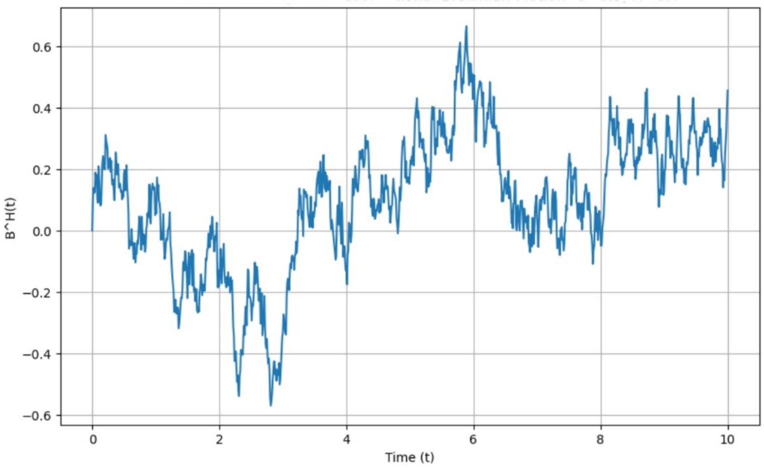

Figure 2: Sample trajectory of a fractional Brownian motion.

1.6 Geometric Fractional Brownian Motion

Geometric fractional Brownian motion (GfBM) is defined as

$$X(t) = X(0)\exp\Big[\Big(\mu - \frac{\sigma^2}{2}\Big)t + \sigma B^H(t)\Big],$$

which ensures that the process remains strictly positive. The model introduces long-range dependence through the Hurst parameter H. In terms of log-returns, GfBM preserves normality but alters the time scaling of the variance from Δt to $(\Delta t)^{2H}$. Figure 3 illustrates a sample trajectory generated with parameters $\mu = 0.7$, $\sigma = 0.8$, $H = 0.4$, and $X(0) = 100$.

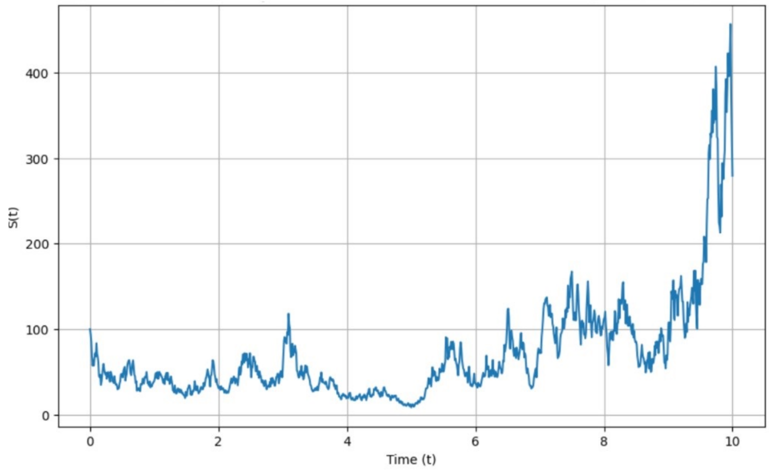

Figure 3: Sample trajectory of a geometric fractional Brownian motion.

1.7 Model Parameter Estimation

1.7.1 Estimating the Hurst Parameter H

Let $Y(t)$ denote either an fBM or the natural logarithm of a GfBM process. For such processes, the variance satisfies

$$\mathbb{V}ar\big(Y(\Delta t)\big) \;=\; \sigma^2(\Delta t)^{2H}.$$

Taking logarithms on both sides yields

$$\ln\big(\mathbb{V}ar(Y(\Delta t))\big) \;=\; 2\ln\sigma \;+\; 2H\ln(\Delta t).$$

A practical method for estimating H is therefore to consider multiple time increments Δt and regress the logarithm of the empirical variance of $Y(\Delta t)$ on $\ln(\Delta t)$. The slope of the resulting regression provides an estimate of twice the Hurst parameter, denoted $2\hat{H}$.

1.7.2 Estimating Volatility σ

The method of moment estimation of the volatility parameter σ depends on the assumed scaling behavior of the variance, which is governed by the Hurst parameter H. Suppose a process is observed on a time grid $0 = t_0, t_1, \ldots, t_n$.

For **GBM** (corresponding to $H = 0.5$), the variance of log-returns grows linearly with time. Accordingly, volatility is estimated using the sample standard deviation of scaled log-returns

$$\frac{\ln\big(X(t_k)/X(t_{k-1})\big)}{\sqrt{t_k - t_{k-1}}}, \quad k = 1, \ldots, n.$$

For **fBM**, the volatility is estimated from the sample standard deviation of price increments scaled using the estimated Hurst exponent:

$$\frac{B^H(t_k) - B^H(t_{k-1})}{(t_k - t_{k-1})^{\hat{H}}}.$$

For **GfBM**, the volatility is estimated from the sample standard deviation of log-returns adjusted using the estimated Hurst parameter:

$$\frac{\ln\big(X(t_k)/X(t_{k-1})\big)}{(t_k - t_{k-1})^{\hat{H}}}.$$

1.7.3 Estimating Drift μ

Log-returns in GBM and GfBM processes have mean $(\mu - \frac{\sigma^2}{2})\,\Delta t$. Thus, for a fixed grid $0 = t_0, t_1, \ldots, t_n$, the drift μ can be estimated by the method of moments:

$$\hat{\mu} = \frac{\hat{\sigma}^2}{2} + \texttt{sample mean of } \frac{\ln\big(X(t_k)/X(t_{k-1})\big)}{t_k - t_{k-1}}, \quad k = 1, \ldots, n.$$

1.8 Model Fit via Kolmogorov–Smirnov Test

The Kolmogorov–Smirnov (KS) test compares the empirical cumulative distribution function of standardized increments to that of a N(0,1) distribution using the test statistic

$$D_n = \sup_x |F_n(x) - F(x)|.$$

Small p-values indicate deviation from normality, while larger p-values indicate that the normality assumption cannot be rejected. Because financial time series exhibit dependence, the KS p-values are used here as a diagnostic rather than a full model validation.The standardization for each model is defined as follows.

In GBM process:

$$\frac{\ln\big(X(t_k)/X(t_{k-1})\big) - \big(\hat{\mu} - \frac{\hat{\sigma}^2}{2}\big)(t_k - t_{k-1})}{\hat{\sigma}\sqrt{t_k - t_{k-1}}} \sim N(0,1).$$

In fBM process:

$$\frac{B^H(t_k) - B^H(t_{k-1})}{\hat{\sigma}(t_k - t_{k-1})^{\hat{H}}} \sim N(0,1).$$

In GfBM process:

$$\frac{\ln\big(X(t_k)/X(t_{k-1})\big) - \big(\hat{\mu} - \frac{\hat{\sigma}^2}{2}\big)(t_k - t_{k-1})}{\hat{\sigma}(t_k - t_{k-1})^{\hat{H}}} \sim N(0,1).$$

2 Applications

2.1 Methodology and Data

In this paper, we analyze three assets to illustrate the different market behaviors that best suit these models. All datasets used were sourced from Yahoo Finance. Parameters such as μ, σ, and H are estimated using the training period, and the goodness of fit of each model is assessed on the testing period using the Kolmogorov–Smirnov test. The three stock prices selected for the analysis correspond to the following companies over the indicated time period:

• **Tesla, Inc. (TSLA)** represents a high-growth technology stock characterized by strong innovation-driven dynamics and high volatility. The analysis uses daily stock price data from June 29, 2010, to December 31, 2024.

• **Consolidated Edison, Inc. (ED)** is a regulated utility company and is typically viewed as a defensive stock. Its price dynamics tend to be more stable, with lower volatility and smoother returns. Daily stock price data from January 1, 2018, to December 31, 2023, are considered.

• **The Sherwin-Williams Company (SHW)** operates in the industrial and consumer goods sector and exhibits intermediate behavior between growth-oriented and defensive assets. The analysis is based on daily stock price data from January 1, 2016, to December 31, 2019.

The time periods for each stock are chosen to capture representative market conditions relevant to the characteristics of each asset.

2.2 Results of Model Fitting

The datasets are divided into training and testing subsets. The GBM, fBM, and GfBM models are fitted using the training data, and the resulting parameter estimates are summarized in Table 1.

Table 1: Parameter estimates over training period.

Stock	**Training Period**	**GBM** $\hat{\mu}, \hat{\sigma}$	**fBM** $\hat{\sigma}, \hat{H}$	**GfBM** $\hat{\mu}, \hat{\sigma}, \hat{H}$
TSLA	2010-2019	0.4325, 0.5120	5.6629, 0.4637	0.4325, 0.4770, 0.4884
ED	2018-2022	0.0931, 0.2435	12.7606, 0.4473	0.0931, 0.1764, 0.4338
SHW	2016-2018	0.1792, 0.2168	22.7151, 0.5080	0.1792, 0.1944, 0.4881

For each fitted model, goodness-of-fit is assessed using the Kolmogorov–Smirnov test, with the corresponding p-values reported in Table 2.

Table 2: Kolmogorov-Smirnov test p-values over testing period.

stock	**p-value (GBM)**	**p-value (fBM)**	**p-value (GfBM)**
TSLA	**0.052146**	0.000000	0.041999
ED	0.003898	**0.090964**	0.001447
SHW	0.209960	0.008036	**0.379900**

Note: Bold values indicate the best-fitting model for that scenario.

Our analysis indicates that the TSLA stock price dynamics are most appropriately captured by a geometric Brownian motion (GBM), while the ED stock price is better characterized by a fractional Brownian motion (fBM). In contrast, the SHW stock price exhibits dynamics that are best described by a generalized fractional Brownian motion (GfBM). Figures 4–6 present the empirical price trajectories alongside collections of simulated paths obtained from the corresponding models.

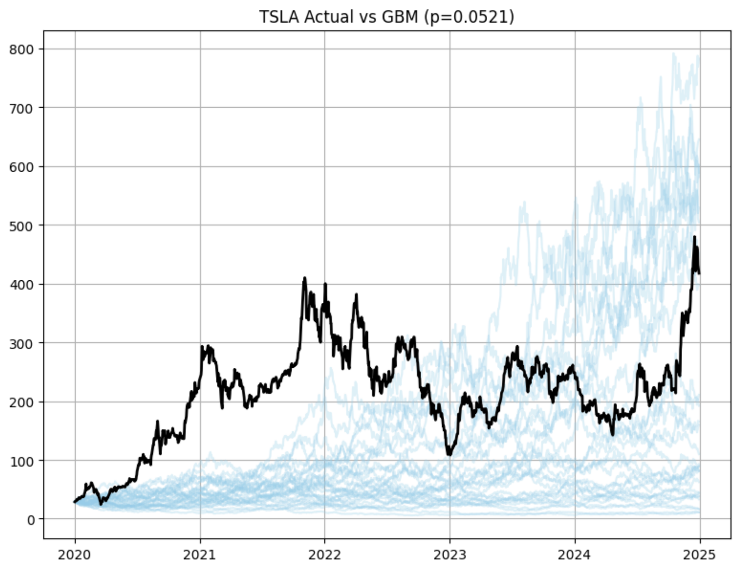

Figure 4: TSLA: Actual price vs. GBM simulation.

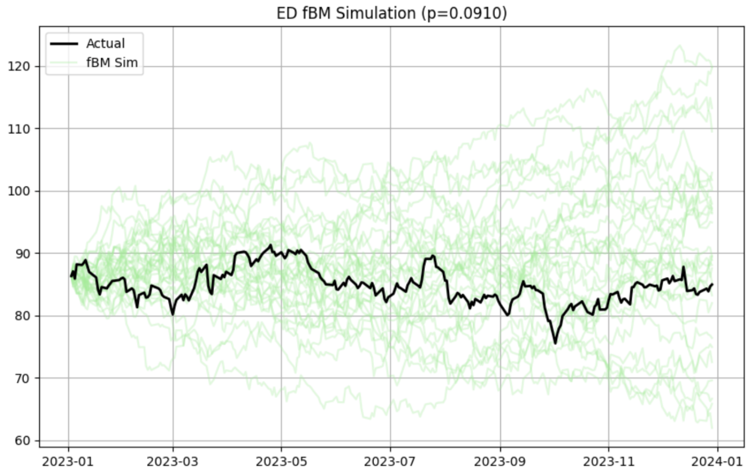

Figure 5: ED: Actual price vs. fBM simulation.

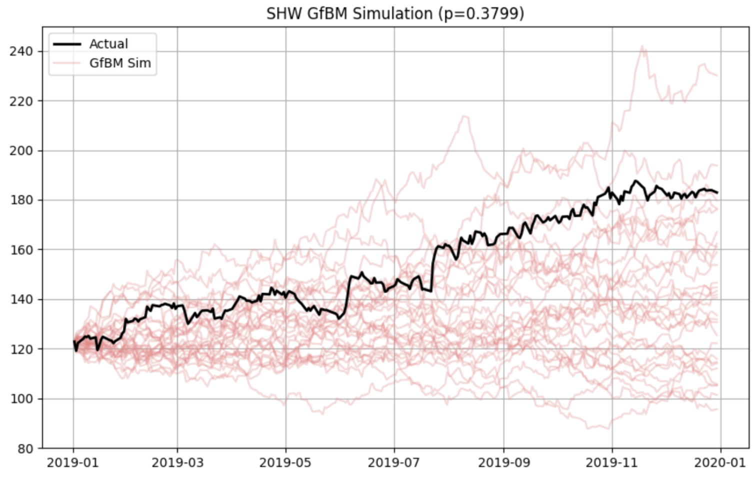

Figure 6: SHW: Actual price vs. GfBM simulation.

3 Conclusion

3.1 Main Findings

This study demonstrates that there is no single "best" way to model a stock; rather, different models are appropriate depending on the underlying behavior of the asset.

- **GBM (Tesla):** Tesla proved to be a strong fit for GBM, suggesting that its price movements during the testing period were highly random, similar to a "random walk." Despite the stock's high volatility, the price changes did not show significant long-term dependence, meaning the standard GBM assumption of independent increments was sufficient to capture its rapid growth.

- **fBM (Con Edison):** As a regulated utility, Con Edison is much more stable and tends to revert to a mean rather than grow exponentially. This behavior is well captured by fBM, which models arithmetic changes rather than geometric ones. The model worked best here because the stock price remained within a relatively consistent range, allowing the Hurst parameter to account for the stock's stability.

- **GfBM (Sherwin-Williams):** Sherwin-Williams represents a middle ground between the high-growth tech sector and the stable utility sector. The superior fit of GfBM indicates that this stock requires a dual approach: the geometric component captures the compounding growth over time, while the fractional component captures the trends and patterns in the price data that a simple GBM would miss.

3.2 Limitations and Future Steps

Although this study supports the relevance of fractional and geometric models, several limitations remain. One key limitation is the small number of stocks analyzed. While the models captured general behavior across all three cases, real-world stock dynamics are considerably more complex and cannot be reduced to a small set of simple categories. For instance, the fBM model estimates an arithmetic volatility parameter. In the training period, TSLA's average price was $23 with a volatility of 5.66. However, as TSLA's price rose to $200 in the testing period, this fixed arithmetic volatility became negligible relative to the stock price, causing the model to underestimate the magnitude of price swings.

Future work should prioritize:

- **Stochastic Volatility:** Incorporating Heston or GARCH-type components to better capture the leptokurtic ("fat-tailed") return distributions observed in the TSLA dataset.

- **Rough Volatility:** Exploring models in which volatility itself is driven by a fractional process with $H < 0.5$, an area of active and rapidly growing research in modern quantitative finance.

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Supplemental Materials

All datasets used in this study are publicly available at the authors' GitHub repository: https://github.com/Anish-Varada/Brownian-Motion-in-Financial-Assets/tree/main.

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