FROM ANALYSIS TO APPLICATION: UNDERSTANDING THE EULER-MACLAURIN FORMULA

ISAAC SUN

Abstract. In this paper we introduce and detail the Euler-Maclaurin Formula which relates summations to their integral counterparts. We start with the Basel Problem which inspired Euler to derive such a formula, delve into an inductive proof of the formula, and then explore Euler's constant and other applications of this formula.

1. Introduction

Our story begins with the Basel Problem, which although quite simple to state, had stumped mathematicians for over 90 years. Euler solved the problem but it left him unsatisfied; he had to find a general way to approximate sums like the Basel Problem. Thus he later derived the Euler-Maclaurin formula in 1732, which was subsequently derived by Maclaurin in 1742. However, when deriving the Euler-Maclaurin Formula, both Euler and Maclaurin were unable to solve for the exact remainder term and it wasn't until 1823 when Poisson discovered it.

The formula can be used to approximate finite sums and infinite series by their integral counterparts and conversely approximate integrals by finite sums. More specifically, an estimation of $\sum_{i=m}^{n} f(i)$ can be found through the integral $\int_m^n f(t)dt$ with an error term that can be expressed through the Bernoulli numbers and a remainder term expressed through the Bernoulli Periodic Functions. Euler derived this formula to approximate many of the converging infinite series that he solved [3]. In its general form, it can be written as:

$$\sum_{n=a}^{b} f(n) = \int_a^b f(t)dt + \sum_{r=0}^{k} \frac{(-1)^{r+1} B_{r+1}}{(r+1)!} \left(f^{(r)}(b) - f^{(r)}(a)\right) + R_k$$

where

$$R_k = \frac{(-1)^k}{(k+1)!} \int_a^b B_{k+1}(t) f^{(k+1)}(t)dt,$$

under the condition that $f(x)$ is k times differentiable on the interval $[m, n]$ for all integers $k \geq 0$. The R_k represents the remainder term and we'll go more in-depth into its notation later.

2. Basel Problem

The Basel Problem asks for the sum of the reciprocals of the squares of the positive integers, or

$$\sum_{n=1}^{\infty} \frac{1}{n^2} =?$$

We can quickly test that it converges by showing that

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\frac{1}{8^2}+\dots$$

is less than

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{4^2}+\frac{1}{4^2}+\frac{1}{4^2}+\frac{1}{8^2}+\dots$$

which converges to 2.

Here Euler decided to express the coefficient of x^3 in the infinite expansion of $\sin x$ in 2 ways. First, he expressed $\sin x$ as the Maclaurin series

$$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

and clearly the coefficient of x^3 is $-\frac{1}{6}$.
Next, he used the Weirstrass factorization to express

$$\sin x = ax(x+\pi)(x-\pi)(x+2\pi)(x-2\pi)\dots = ax\prod_{1}^{\infty}(x^2-(n\pi)^2).$$

This implies that

$$a=\frac{\sin x}{x}\frac{1}{(x^2-\pi^2)(x^2-(2\pi)^2)(x^2-(3\pi)^2)\dots},$$

, and combined with $\lim_{x\to 0}\frac{\sin x}{x}=1$,

$$a=\frac{1}{[-(\pi)^2][-(2\pi)^2][-(3\pi)^2)]\dots}.$$

After expansion, our result for the x^3 coefficient is

$$-\sum_{n=1}^{\infty}\frac{1}{(n\pi)^2}.$$

Setting this equal to $-\frac{1}{6}$ with a bit of rearranging, we conclude that

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$$

With this, Euler had solved the Basel Problem, and this gave him his biggest inspiration for the Euler-Maclaurin formula.

3. Preliminaries

To understand the Euler-Maclaurin Summation Formula, we must first define the Bernoulli numbers b_n and the Bernoulli polynomials $B_n(x)$. They both occur in a number of different theorems involving analysis and number theory.

3.1. **Bernoulli Numbers.** The Bernoulli numbers b_n can be defined by the following power series:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}, \quad \text{where}$$

or

$$b_n = \frac{d^n}{dx^n}\left(\frac{x}{e^x - 1}\right)\Bigg|_{x=0}.$$

Here are the first few coefficients b_n:

n	b_n
0	1
1	-$\frac{1}{2}$
2	$\frac{1}{6}$
3	0
4	$-\frac{1}{30}$
5	0
6	$\frac{1}{42}$

Notice how for any integer $k > 1$, $b_{2k-1} = 0$, and how b_n is negative if divisible by 4, and positive otherwise. This is trivialized by the Taylor expansion of $\frac{x}{e^x-1}$. There happens to be no simple pattern to the Bernoulli numbers, but a good approximation is

$$b_{2n} \approx (-1)^{n-1} 4\sqrt{\pi n}\left(\frac{n}{\pi e}\right)^{2n}.$$

3.2. **Bernoulli Polynomials.** In a similar fashion, we can define the Bernoulli Polynomial for nonnegative integers n as such:

$$\frac{ze^{zx}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!}.$$

There are 4 important properties about the Bernoulli Polynomials.

Endpoint Property:

$$B_n(1) = \sum_{j=0}^{n} \binom{n}{j} B_j = B_n = B_n(0), \quad n \geq 2.$$

Differentiation:

$$B_n'(t) = \sum_{j=0}^{n-1} (n-j) \binom{n}{j} B_j t^{n-j-1} = nB_{n-1}(t), \quad n \geq 1.$$

Integration:

$$\int_0^1 B_n(t)dt = \frac{1}{n+1} \int_0^1 B_{n+1}'(t)dt = 0, \quad n \geq 1.$$

Upper Bound:

$$|B_{2r}(t)| \leq |B_{2r}|, \quad r \geq 0.$$

A sketch of the Upper Bound is as follows:

Proof. To prove this upper bound, let's first define the Periodic Bernoulli Function. Now define the Periodic Bernoulli Function as such, where $\{x\} = x - \lfloor x \rfloor$:

$$P_n(x) = B_n(\{x\}).$$

We can express the remainder term R_p as

$$R_k = \frac{(-1)^k}{(k+1)!} \int_a^b B_{k+1}(t) f^{(k+1)}(t) dt,$$

Now let's begin by repeatedly differentiating our Bernoulli polynomial. This gives

$$B_n^{(j)} = \frac{n!}{(n-j)!} B_{n-j}(t), \quad j = 0, 1, \ldots, n, \quad n \geq 1,$$

implying

$$B_n^{(j)}(0) = B_n^{(j)}(1), \quad j = 0, 1, \ldots, n-2.$$

Therefore, $P_n \in C^{n-2}(\mathbb{R})$. Now let's consider the Fourier series of P_n,

$$P_n(t) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k t}.$$

Since $P_n(t)$ is real, $c_{-k} = \bar{c_k}$. Letting $c_k = a_k + ib_k$, we have $a_{-k} = a_k$ and $b_{-k} = -b_k$. Thus,

$$P_n(t) = a_0 + 2\sum_{k=1}^{\infty} a_k \cos(2\pi i k t) + 2\sum_{k=1}^{\infty} b_k \sin(2\pi i k t).$$

The coefficients are

$$c_k = \int_0^1 B_n(t) e^{-2\pi i k t} dt.$$

For $k = 0$,

$$c_0 = \int_0^1 B_n(t) dt = 0, \quad n \geq 1,$$

and for other k we get

$$c_k = [B_n(t) \frac{e^{-2\pi i k t}}{-2\pi i k}]_0^1 + n \int_0^1 B_{n-1}(t) e^{-2\pi i k t} dt = \frac{-n!}{(2\pi i k)^n} = a_k + ib_k.$$

Thus, $a_0 = b_0 = 0$ and for values of $k \neq 0$ we have different formulas for a_k and b_k as follows:

$$a_k = (-1)^{r-1} \frac{(2r)!}{(2\pi k)^{2r}}, b_k = 0, \quad \text{for} \quad n = 2r,$$

and

$$a_k = 0, b_k = (-1)^r \frac{(2r-1)!}{(2\pi k)^{2r-1}}, \quad \text{for} \quad n = 2r-1.$$

From this, we can deduce that

$$P_{2r}(t) = (-1)^{r-1} 2(2r)! \sum_{k=1}^{\infty} \frac{\cos(2\pi kt)}{(2\pi k)^{2r}},$$

and

$$P_{2r-1}(t) = (-1)^{r} 2(2r-1)! \sum_{k=1}^{\infty} \frac{\sin(2\pi kt)}{(2\pi k)^{2r-1}}.$$

We conclude the proof by piecing it all together as follows:

$$|B_{2r}(t)| \leq 2(2r)! \sum_{k=1}^{\infty} \frac{1}{(2\pi k)^{2r}} = |P_{2r}(0)| = |B_{2r}|.$$

$\square$

4. Proof of the Euler-Maclaurin Formula

4.1. **Proof 1.** We proceed with the proof by induction. To prove the formula for $k = 0$, we first rewrite $\int_{n-1}^{n} f(t)dt$, where n is an integer, using integration by parts. We obtain

$$\begin{aligned} \int_{n-1}^{n} f(t)dt = \int_{n-1}^{n} \frac{d}{dt}\left(t - n + \frac{1}{2}\right) f(t)dt &= \left(t - n + \frac{1}{2}\right) f(t)\Big|_{n-1}^{n} - \int_{n-1}^{n} \left(t - n + \frac{1}{2}\right) f'(t)dt \\ &= \frac{1}{2}(f(n) + f(n-1)) - \int_{n-1}^{n} \left(t - n + \frac{1}{2}\right) f'(t)dt. \end{aligned}$$

Because $t - n + \frac{1}{2} = B_1(t)$ on the interval $(n-1, n)$, this is equal to

$$\int_{n-1}^{n} f(t)dt = \frac{1}{2}(f(n) + f(n-1)) - \int_{n-1}^{n} B_1(t) f'(t)dt$$

From this, we get

$$f(n) = \int_{n-1}^{n} f(t)dt + \frac{1}{2}(f(n) - f(n-1)) + \int_{n-1}^{n} B_1(t) f'(t)dt.$$

Now we take the sum of this expression for $n = a+1, a+2, \ldots, b$, so that the middle term on the right telescopes away for the most part:

$$\sum_{n=a+1}^{b} f(n) = \int_{a}^{b} f(t)dt + \frac{1}{2}(f(b) - f(a)) + \int_{a}^{b} B_1(t) f'(t)dt,$$

which is the Euler-Maclaurin formula for $k = 0$, since $B_1 = -\frac{1}{2}$. Suppose that $k > 0$ and the formula is correct for $k - 1$, that is

$$\sum_{a<n\leq b} f(n) = \int_{a}^{b} f(t)dt + \sum_{r=0}^{k-1} \frac{(-1)^{r+1} B_{r+1}}{(r+1)!} \left(f^{(r)}(b) - f^{(r)}(a)\right) + \frac{(-1)^{k-1}}{k!} \int_{a}^{b} B_k(t) f^{(k)}(t)dt.$$

We rewrite the last integral using integration by parts, the fact that B_k is continuous for $k \geq 2$, and that $B'_{k+1}(t) = (k+1)B_k(t)$ for $k \geq 0$:

$$\begin{aligned}\int_a^b B_k(t) f^{(k)}(t)dt &= \int_a^b \frac{B'_{k+1}(t)}{k+1} f^{(k)}(t)dt \\ &= \frac{1}{k+1} B_{k+1}(t) f^{(k)}(t)\Big|_a^b - \frac{1}{k+1}\int_a^b B_{k+1}(t) f^{(k+1)}(t)dt.\end{aligned}$$

Using the fact that $B_k(n) = B_k$ for every integer n if $k \geq 2$, we see that the last term is equal to

$$\frac{(-1)^{k+1} B_{k+1}}{(k+1)!}\left(f^{(k)}(b) - f^{(k)}(a)\right) + \frac{(-1)^k}{(k+1)!}\int_a^b B_{k+1}(t) f^{(k+1)}(t)dt.$$

Substituting this and absorbing the left term into the summation yields

$$\sum_{n=a}^{b} f(n) = \int_a^b f(t)dt + \sum_{r=0}^{k} \frac{(-1)^{r+1} B_{r+1}}{(r+1)!}\left(f^{(r)}(b) - f^{(r)}(a)\right) + R_k.$$

4.2. **Proof 2.** Another way to prove the Euler-Maclaurin formula is by first showing that the local version is true, and then expanding to the global version.

Lemma 4.1. *For $r \geq 0$ and $F \in C^{2r+2}[0,1]$,*

$$\int_0^1 F(t)dt = \frac{1}{2}(F(0) + F(1)) - \sum_{k=1}^{r+1} \frac{B_{2k}}{(2k)!}\left(F^{(2k-1)}(1) - F^{(2k-1)}(0)\right) + R_r$$

where

$$R_r = \frac{1}{(2r+2)!}\int_0^1 B_{2r+2}(t) F^{(2r+2)}(t)dt$$

Proof. By applying integration by parts twice, we obtain

$$\begin{aligned}\int_0^1 F(t)dt &= \int_0^1 B_0(t)F(t)dt = [B_1(t)F(t)]_0^1 - \int_0^1 B_1(t)F'(t)dt \\ &= \frac{1}{2}(F(0) + F(1)) - \frac{1}{2}\left[B_2(t)F'(t)\right]_0^1 + \frac{1}{2}\int_0^1 B_2(t)F''(t)dt \\ &= \frac{1}{2}(F(0) + F(1)) - \frac{B_2}{2}\left(F'(1) - F'(0)\right) + R_0,\end{aligned}$$

which proves the formula for the $r = 0$ case. Now suppose $r \geq 1$, and assume that the lemma holds with r replaced by $r - 1$. Applying integration by parts twice, we obtain

$$\begin{aligned}R_{r-1} &= \frac{1}{(2r)!}\int_0^1 B_{2r}(t)F^{(2r)}(t)dt \\ &= \frac{1}{(2r+1)!}\left[B_{2r+1}(t)F^{(2r)}(t)\right]_0^1 - \frac{1}{(2r+1)!}\int_0^1 B_{2r+1}(t)F^{(2r+1)}(t)dt \\ &= -\frac{1}{(2r+2)!}\left[B_{2r+2}(t)F^{(2r+1)}(t)\right]_0^1 + \frac{1}{(2r+2)!}\int_0^1 B_{2r+2}(t)F^{(2r+2)}(t)dt \\ &= -\frac{1}{(2r+2)!}\left(F^{(2r+1)}(1) - F^{(2r+1)}(0)\right) + R_r,\end{aligned}$$

which completes the proof. □

Lemma 4.2. *For $r \geq 0$ and $F \in C^{2r+2}[0,1]$, there is some $\xi \in (0,1)$ such that*

$$\int_0^1 F(t)dt = \frac{1}{2}(F(0)+F(1)) - \sum_{k=1}^{r} \frac{B_{2k}}{(2k)!}\left(F^{(2k-1)}(1) - F^{(2k-1)}(0)\right) - R$$

where

$$R = \frac{B_{2r+2}}{(2r+2)!} F^{(2r+2)}(\xi).$$

Proof. The last term plus the remainder term can be expressed as

$$\begin{aligned} &-\frac{B_{2r+2}}{(2r+2)!}\left(F^{(2r+1)}(1) - F^{(2r+1)}(0)\right) + R_r \\ &= -\frac{B_{2r+2}}{(2r+2)!}\int_0^1 F^{(2r+2)}(t)dt + R_r = -R \end{aligned}$$

where

$$R := \frac{1}{(2r+2)!}\int_0^1 \left(B_{2r+2} - B_{2r+2}(t)\right) F^{(2r+2)}(t)dt$$

Using the upper bound on $B_{2r}(t)$ that was derived in the previous section, we note the difference $B_{2r} - B_{2r}(t)$ is of one sign in $[0,1]$, because

$$\operatorname{sgn}(B_{2r})\left(B_{2r} - B_{2r}(t)\right) = |B_{2r}| - \operatorname{sgn}(B_{2r}) B_{2r}(t) \geq |B_{2r}| - |B_{2r}(t)| \geq 0.$$

So by the Mean Value Theorem, there is some $\xi \in (0,1)$ such that

$$R = \frac{1}{(2r+2)!}\int_0^1 \left(B_{2r+2} - B_{2r+2}(t)\right) dt F^{(2r+2)}(\xi) = \frac{B_{2r+2}}{(2r+2)!} F^{(2r+2)}(\xi).$$

□

Now to finish off our proof, we generalize our local case. Given an interval $[a,b]$, we choose $n \geq 1$ and let $h = (b-a)/n$ and $x_i = a + ih, i = 0, 1, \ldots, n$.

Theorem 4.3. *For $r \geq 0$ and $f \in C^{2r+2}[a,b]$, there is some $\xi \in (a,b)$ such that*

$$\int_a^b f(x)dx = T(h) - \sum_{k=1}^{r} \frac{B_{2k}}{(2k)!} h^{2k}\left(f^{(2k-1)}(b) - f^{(2k-1)}(a)\right) - R$$

where

$$T(h) = \frac{h}{2}(f(a)+f(b)) + h\sum_{i=1}^{n-1} f(x_i)$$

and

$$R = \frac{B_{2r+2}}{(2r+2)!}(b-a)h^{2r+2} f^{(2r+2)}(\xi).$$

Corollary 4.4. *For $r \geq 0$ and $f \in C^{2r+2}[a,b]$,*

$$\int_a^b f(x)dx = T(h) - \sum_{k=1}^{r} \frac{B_{2k}}{(2k)!} h^{2k}\left(f^{(2k-1)}(b) - f^{(2k-1)}(a)\right) + O\left(h^{2r+2}\right),$$

as $h \to 0$.

Proof. Let $F(t) = f(x_{i-1} + ht), t \in [0,1], i = 1, \ldots, n$. Then Lemma 2 gives

$$\int_{x_{i-1}}^{x_i} f(x)dx = h \int_0^1 F(t)dt =$$
$$\frac{h}{2}\left(f(x_{i-1}) + f(x_i)\right) - \sum_{k=1}^{r} \frac{B_{2k}}{(2k)!} h^{2k} \left(f^{(2k-1)}(x_{i-1}) - f^{(2k-1)}(x_i)\right) - R_i$$

where

$$R_i = \frac{B_{2r+2}}{(2r+2)!} h^{2r+3} f^{(2r+2)}(\xi_i)$$

for some $\xi_i \in (x_{i-1}, x_i)$. Summing this equation over $i = 1, \ldots, n$ yields the desired expansion except that the remainder term is

$$R = \sum_{i=1}^{n} R_i = \frac{B_{2r+2}}{(2r+2)!} h^{2r+3} \sum_{i=1}^{n} f^{(2r+2)}(\xi_i).$$

However, an application of the Mean Value Theorem gives

$$\sum_{i=1}^{n} f^{(2r+2)}(\xi_i) = nf^{(2r+2)}(\xi)$$

for some $\xi \in (a, b)$, and thus we are done. □

5. Applications

5.1. **Euler's Constant.** Something interesting happens when we consider the formula for $f(x) = \frac{1}{x}$. Let $a = 1$, $b = n$, and $k = 1$ in the formula. Then the expression becomes:

$$\sum_{i=1}^{n} \frac{1}{i} = \log n + \frac{1}{2n} + \frac{1}{2} + \int_1^n \frac{P_1(x)}{x^2} dx.$$

Now let

$$R(n) = \frac{1}{2n} + \frac{1}{2} + \int_1^n \frac{P_1(x)}{x^2} dx.$$

Note that $P_1(x)$'s absolute value is bounded by $\frac{1}{2}$ and thus $R(n)$ converges when n approaches ∞. Let us denote $\gamma = \lim_{n\to\infty} R(n)$. This γ is actually known as Euler's Constant [1]. A good approximation is

$$\gamma = 0.5772156649015328606065120900....$$

It is unknown whether Euler's Constant is rational or not and it is also believed that it could be transcendental. The question is left as food for thought.

5.2. **Stirling's Formula.** Another interesting application of the formula is the Stirling's approximation formula, which tells us that

Theorem 5.1.

$$n! = C(n)\sqrt{n}\left(\frac{n}{e}\right)^n [4]$$

Proof. Let us take $f(x) = \log x, k = 0, a = 1$, and $b = n$, where n is any positive integer. We have

$$\sum_{i=1}^{n} \log i = (n \log n - n + 1) + \frac{\log n}{2} + \int_1^n \frac{P_1(t)}{t} dt.$$

Simplifying and extracting the error term $R(n)$, we get

$$\begin{aligned} \log n! &= \left(n + \tfrac{1}{2}\right) \log n - (n-1) + \textstyle\int_1^n \frac{P_1(t)}{t} dt \\ R(n) &= \log n! - \left(n + \tfrac{1}{2}\right) \log n + n = \textstyle\int_1^n \frac{P_1(t)}{t} dt + 1 \end{aligned}$$

Integration by parts gives

$$\int_1^n \frac{P_1(t)}{t} dt = \frac{P_2(t)}{t}\bigg|_1^n + \int_1^n \frac{P_2(t)}{t^2} dt.$$

But $P_2(x) = \sum_{i=1}^{\infty} \frac{2\cos(2i\pi x)}{(2i\pi)^2}$, so using that the absolute value of cosine function is at most 1 and the series $\sum_{i=1}^{\infty} \frac{1}{i^2}$ converges, we see that absolute value of $P_2(x)$ is bounded. Then the integral above converges when n tends to infinity. Define

$$C = \lim_{n\to\infty} R(n).$$

We can obtain the exact value of C in the following way. We have

$$\begin{aligned} 2\log(2 \cdot 4 \cdots 2n) &= 2n \log 2 + 2 \log n! \\ &= 2n \log 2 + (2n+1) \log n - 2n + 2R(n) \\ &= (2n+1) \log 2n - 2n - \log 2 + 2R(n).z \end{aligned}$$

On the other hand,

$$\log(2n+1)! = \left(2n + \frac{3}{2}\right) \log(2n+1) - (2n+1) + R(2n+1).$$

Subtracting the second expression from the first, we get

$$\begin{aligned} \log \frac{2 \cdot 4 \cdots 2n}{1 \cdot 3 \cdots (2n+1)} &= (2n+1) \log \frac{2n}{2n+1} - \frac{1}{2} \log(2n+1) + 1 - \log 2 + 2R(n) - R(2n+1) \\ &= -\log \left(1 + \frac{1}{2n}\right)^{2n+1} - \frac{1}{2} \log(2n+1) + 1 - \log 2 + 2R(n) - R(2n+1). \end{aligned}$$

Now let's apply Walli's product formula.

Theorem 5.2.

$$\lim_{n\to\infty} \frac{2 \cdot 4 \cdots 2n}{1 \cdot 3 \cdots (2n+1)} \frac{1}{\sqrt{n}} = \sqrt{\pi}.$$

Using the continuity of the logarithmic function in the formula above, we get

$\lim_{n\to\infty} \log \frac{2\cdot4\cdots2n}{1\cdot3\cdots(2n-1)} \frac{1}{\sqrt{2n+1}} = -\lim_{n\to\infty} \log \left(1 + \frac{1}{2n}\right)^{2n+1} + 1 - \log 2 + \lim_{n\to\infty}(2R(n) - R(2n+1))$

which we can rewrite as

$$\log \frac{1}{\sqrt{2}} \lim_{n\to\infty} \frac{2 \cdot 4 \cdots 2n}{1 \cdot 3 \cdots (2n+1)} \frac{1}{\sqrt{n}} = C - \log 2.$$

Thus, we have $C = \log \sqrt{2\pi}$. Substituting back into the remainder term above finishes our proof. $\square$

5.3. **Superconvergence of the Trapezoidal Rule.**

Corollary 5.3. *Suppose $r \geq 0$ and $f \in C^{2r+2}[a,b]$. If $f^{(2k-1)}(b) = f^{(2k-1)}(a)$ for $k = 1, \ldots, r$, then*

$$\int_a^b f(x)dx = T(h) + O\left(h^{2r+2}\right) \text{ as } h \to 0.$$

This will be the case for any $r \geq 0$ for functions $f \in C^\infty(\mathbb{R})$ that are periodic with period $b - a$. In fact, for some functions of this type, the trapezoidal rule is exact.

Theorem 5.4. *Let $[a,b] = [0, 2\pi]$ and let*

$$f(x) = \sum_{k=0}^{n-1} a_k \cos(kx) + \sum_{k=1}^{n-1} b_k \sin(kx),$$

for any choice of a_k and b_k in $\mathbb{R}$. Then $T(h)$ is exact for f.

Proof. All we need to show is that $T(h)$ is exact for $f(x) = e^{ikx}, k = 0, 1, \ldots, n-1$. The integral of f is

$$\int_0^{2\pi} f(x)dx = \begin{cases} 2\pi, & k = 0, \\ 0, & k > 0 \end{cases}$$

On the other hand, since $f(0) = f(2\pi)$,

$$\begin{aligned} T(h) &= h\left(\frac{1}{2}(f(0) + f(2\pi)) + \sum_{j=1}^{n-1} f\left(\frac{2j\pi}{n}\right)\right) \\ &= h\sum_{j=0}^{n-1} f\left(\frac{2j\pi}{n}\right) = h\sum_{j=0}^{n-1} e^{2jki\pi/n}. \end{aligned}$$

So, if $k = 0$,

$$T(h) = hn = 2\pi,$$

and if $1 \leq k \leq n-1$,

$$T(h) = h\sum_{j=0}^{n-1}\left(e^{2ki\pi/n}\right)^j = h\frac{e^{2ki\pi} - 1}{e^{2ki\pi/n} - 1} = 0,$$

and thus, we are done. [5] ∎

5.4. **Sum of p-th Powers.** A neat application is a formula for sums of p-th powers.

Corollary 5.5. *For $p \geq 1$,*

$$\sum_{j=1}^{n-1} j^p = \frac{1}{p+1}\left(B_{p+1}(n) - B_{p+1}\right).$$

The first few examples are as follows:

$$\sum_{j=1}^{n-1} j = \frac{1}{2}n^2 - \frac{1}{2}n,$$
$$\sum_{j=1}^{n-1} j^2 = \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n,$$
$$\sum_{j=1}^{n-1} j^3 = \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{1}{4}n^2,$$
$$\sum_{j=1}^{n-1} j^4 = \frac{1}{5}n^5 - \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n,$$

Proof. Let $f(x) = x^p, p \geq 1$, and $[a, b] = [0, n]$. Then

$$\int_a^b f(x) = \int_0^n x^p dx = \frac{n^{p+1}}{p+1}.$$

With $h = 1$, applying the trapezoidal rule for f on $[0, n]$ gives

$$T(h) = \frac{1}{2}n^p + \sum_{j=1}^{n-1} j^p.$$

Let r be such that $p = 2r$ or $p = 2r + 1$. Then applying the Euler-Maclaurin Formula, we get

$$\int_0^n x^p dx = T(h) - \sum_{k=1}^{r} \frac{B_{2k}}{(2k)!} \frac{p!}{(p-2k+1)!} n^{p-2k+1},$$

and therefore,

$$\begin{aligned}
\sum_{j=1}^{n-1} j^p &= \frac{n^{p+1}}{p+1} - \frac{1}{2}n^p + \frac{1}{p+1} \sum_{k=1}^{r} \binom{p+1}{2k} B_{2k} n^{p-2k+1} \\
&= \frac{1}{p+1} \sum_{k=0}^{p} \binom{p+1}{k} B_k n^{p-k+1} \\
&= \frac{1}{p+1} \left(B_{p+1}(n) - B_{p+1} \right).[6]
\end{aligned}$$

□

Acknowledgements

The author would like to thank Dr. Simon Rubinstein-Salzedo for providing the opportunity to write this paper, as well as Eric Frankel for his support and expertise on the topic.

References

[1] Leonhard Euler. "Concerning the Summation of Innumerable Progressions". In: (1730), pp. 1–13. URL: `http://www.17centurymaths.com/contents/euler/e020tr.pdf`.

[2] Leonhard Euler. "Concerning the Summation of Innumerable Progressions". In: (1732), pp. 1–26. URL: `http://www.17centurymaths.com/contents/euler/e025tr.pdf`.

[3] Michael Floater. "Euler-Maclaurin formula". In: (2019), pp. 5–11. URL: `https://www.uio.no/studier/emner/matnat/math/MAT4130/v19/undervisningsmateriale/euler-maclaurin.pdf`.

[4] Victor Kac and Pokman Cheung. "Euler-Maclaurin Formula". In: *Quantum Calculus*. Springer, 2002.

[5] Stella Mills. "The independent derivations by Leonhard Euler and Colin Maclaurin of the Euler-Maclaurin summation formula". In: *Archive for history of exact sciences* 33.1/3 (1985), pp. 1–13.

[6] Michael Z Spivey. "The Euler-Maclaurin formula and sums of powers". In: *Mathematics magazine* 79.1 (2006), pp. 61–65.